Evaluation of Optimized CPR Schemes for Computational Aeroacoustics Benchmark Problems

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Correction Procedure via Reconstruction (CPR) is a differential discontinuous formulation for conservation laws. In this paper, a recently developed optimized CPR scheme, named frequency optimized CPR (FOCPR), was employed and tested. The idea is to resolve the broadest range of wave frequencies on a given mesh given an acceptable error threshold. In the FOCPR method, both polynomial and Fourier components are included in the basis. The free-parameters of the Fourier basis were chosen to minimize both dispersion and dissipation errors. The performance of the CPR and the FOCPR schemes was compared with a set of CAA problems: the 1D linear wave propagation, the 1D wave propagation on a 2D mesh governed by the 2D Euler equations, the Fourth Computational Aeroacoustics (CAA) Workshop on Benchmark Problems: Category 2 (C2) and Problem 2 Category 3 (C3P2). It was shown that the FOCPR scheme was able to give more accurate solutions than the CPR scheme for CAA problems on meshes with barely enough resolution for high-frequency waves. For problems with widely varying mesh sizes and non-dimensional wave numbers, the benefit from the FOCPR scheme may be limited.

I. Introduction

High-order methods have received much attention for their ability to achieve high accuracy on relatively coarse meshes. In the last two decades, many powerful high-order numerical methods capable of handling unstructured meshes have been developed, e.g. the spectral element method, k-exact finite volume method, discontinuous Galerkin (DG) method, spectral volume, spectral difference and the correction procedure via reconstruction (CPR) methods.

The CPR method was recently developed in [4], and extended to simplex meshes in [17]. Further developments have been described in [26,27,28]. The degrees-of-freedom (DOFs) are the state variables of a pre-defined nodal set named solution points (SPs), where the differential form of the governing equations is solved. As a result, explicit surface and volume integrals are avoided. The CPR formulation is among the most efficient discontinuous methods in terms of the number of operations.

The stability and accuracy of the CPR method depend on the choice of the solution approximation and the weighting functions. Generally, the piecewise polynomial space is chosen for convection problems. However, they may not provide the best approximation for some PDEs and initial/boundary conditions. Here are some examples in the literature. The locally divergence-free polynomial space was used in the DG method to solve the Maxwell equations and better results were achieved compared to the classical piecewise polynomial space in [1,8,9,10]. Exponential functions were proposed to solve singular perturbation problems by Kadalbajoo and Patidar [5] and by Reddy and Chakravarthy [11]. Non-polynomial spaces were used in the local essentially non-oscillatory (ENO) reconstruction for solving hyperbolic conservation laws in [2]. Exponential functions were also used near a boundary, and the trigonometric functions for highly oscillatory problems, as shown by Yuan and Shu [23].

Computational aeroacoustics problems impose stringent requirements on the accuracy of the numerical algorithms because the acoustics disturbance amplitudes are several orders of magnitude smaller than the mean flow disturbance. Because of its high accuracy, the CPR method is a good candidate for CAA problems. However, the high-frequency components of broadband waves can be severely damped due to the large dispersion and dissipation errors of the solution algorithm. Resolving high-frequency waves remains a difficult challenge in the development of computational aeroacoustics (CAA) algorithms.

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Recently, a hybrid basis including both polynomial and Fourier terms was employed to resolve broadband wave propagation problems. It borrows the idea from the dispersion-relation-preserving (DRP) method [6,12,14,15,24,25] to minimize both the dispersion and dissipation errors. In the DRP method, the scheme coefficients are optimized for the high resolution of high frequency waves with respect to the computation grid instead of the truncation errors. To achieve a similar result, in the CPR method, Fourier terms were introduced to the basis because they have the ability to exactly represent waves at certain wave numbers, while monomials were employed to preserve a certain order of accuracy [7]. This scheme is referred to as frequency optimized CPR formulation (FOCPR) in this paper.

In this paper, the accuracy and stability properties of the CPR and the FOCPR methods were compared. Fourier analysis of the 4th order CPR scheme and 4 DOFs-per-cell FOCPR scheme is described. In the numerical tests, both the CPR and FOCPR methods were evaluated for 1D and 2D linear wave propagation and two categories of problems in the Fourth Computational Aeroacoustics Workshop on Benchmark Problems [3]: Category 2 (C2) and Problem 2 Category 3 (C3P2). The benefits of the FOCPR method are demonstrated for most cases.

The paper is organized as follows. In the next section, the basic formulation of the CPR and the FOCPR is briefly reviewed. In Section 3, the solutions of the CPR and the FOCPR methods for the aeroacoustic problems are presented and discussed. The conclusions are summarized in Section 4.

II. Review of the CPR and the optimized CPR formulations

A. Review of the CPR formulation

The CPR formulation can be derived from a weighted residual method by transforming the integral formulation into a differential one. The hyperbolic conservation law can be written as

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{F}(u) = 0,$$

(1)

with proper initial and boundary conditions, where \(u\) is the state vector, and \(\mathbf{F}\) is the flux vector. The computation domain \(A\) is discretized into \(N\) non-overlapping elements \(\{V_i\}_{i=1}^N\). Multiplying Eq. (1) with an arbitrary weighting function \(W\) and integrating over an element \(V_i\), we obtain

$$\int_{V_i} \left( \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{F}(u) \right) WdV = \int_{\partial V_i} W \frac{\partial u}{\partial t} dV + \int_{\partial V_i} W \mathbf{F}(u) \cdot \mathbf{n} ds - \int_{V_i} \nabla W \cdot \mathbf{F}(u) dV = 0. \tag{2}$$

Let \(u_i\) be an approximate solution of the analytical solution \(u\) on element \(V_i\). The solution is discontinuous across each element interface. On each element, the solution belongs to the space of polynomials of degree \(k\) or less, i.e \(u_i \in P^k(V_i)\). In addition, the numerical solution \(u_i\) is required to satisfy

$$\int_{V_i} \frac{\partial u_i}{\partial t} WdV + \int_{\partial V_i} W F_{\text{com}}^n(u_i, u_{i+}, \mathbf{n}) dS - \int_{V_i} \nabla W \cdot \mathbf{F}(u_i) dV = 0, \tag{3}$$

where \(F_{\text{com}}^n(u_i, u_{i+}, \mathbf{n})\) is the common Riemann flux, \(u_{i+}\) denotes the solution outside the current element \(V_i\).

Applying integration by parts again to the last term of the above LHS, we obtain

$$\int_{V_i} \frac{\partial u_i}{\partial t} WdV + \int_{V_i} W \nabla \cdot \mathbf{F}(u_i) dV + \int_{\partial V_i} W [F_{\text{com}}^n(u_i, u_{i+}, \mathbf{n}) - F^n(u_i)] dS = 0, \tag{4}$$

where \(F^n(u_i)\) is the normal flux based on the current solution \(u_i\). With the “lifting operator”, the boundary integral above is cast as a volume integral via the introduction of a “correction field” on \(V_i\), \(\delta_i \in P^k(V_i)\),

$$\int_{V_i} W \delta_i dV = \int_{\partial V_i} W[F^n] dS, \tag{5}$$

where \([F^n] = F_{\text{com}}^n(u_i, u_{i+}, \mathbf{n}) - F^n(u_i)\) is the normal flux difference. Substituting Eq.(5) into Eq.(4), we obtain

$$\int_{V_i} \left[ \frac{\partial u_i}{\partial t} + \nabla \cdot \mathbf{F}(u_i) + \delta_i \right] WdV = 0. \tag{6}$$

Eq. (6) is equivalent to

$$\frac{\partial u_i}{\partial t} + \Pi(\nabla \cdot \mathbf{F}(u_i)) + \delta_i = 0, \tag{7}$$

where the \(\Pi(\nabla \cdot \mathbf{F}(u_i))\) is a projection of \(\nabla \cdot \mathbf{F}(u_i)\) to \(P^k\). Next, let the DOFs be the solutions at a set of solution points (SPs) \(\{\vec{r}_{ij}\}\) (j varies from 1 to K). Then Eq. (7) holds true at the SPs, i.e,
\[
\frac{\partial u_{i,j}}{\partial t} + \Pi_j (\nabla \cdot \vec{F}(u_{i,j})) + \delta_{i,j} = 0,
\]

where \( \Pi_j (\nabla \cdot \vec{F}(u_{i,j})) \) denotes the values of \( \Pi (\nabla \cdot \vec{F}(u_i)) \) at SP j. For linear triangles with straight edges, once the solution points and flux points are chosen, the correction at the SPs can be written as

\[
\delta_{i,j} = \frac{1}{|V_l|} \sum_{f \in V_l} \sum_{i} \alpha_{f,i} [F^n]_{f,i} S_f,
\]

where \( \alpha_{f,i} \) are lifting constants independent of the solution, \( S_f \) is the face area, \( |V_l| \) is the volume of \( V_l \). Substituting Eq. (9) into Eq. (8) we obtain the following CPR formulation

\[
\frac{\partial u_{i,j}}{\partial t} + \Pi_j (\nabla \cdot \vec{F}(u_i)) + \frac{1}{|V_l|} \sum_{f \in V_l} \sum_{i} \alpha_{f,i} [F^n]_{f,i} S_f = 0.
\]

The 1D CPR formulation can be deduced from Eq.(10) as

\[
\frac{\partial u_{i,j}}{\partial t} + \Pi_j \left( \frac{\partial F(u_i)}{\partial x} \right) + \frac{2}{|\Delta x_i|} \left( \alpha_{R,i} [F^n]_{R} + \alpha_{L,i} [F^n]_{L} \right) = 0,
\]

where \( \Delta x_i \) is the length of element \( i \), which has two interfaces, the left one and right one, with unit face areas and unit face normals of -1 and 1. It is often more convenient to transform the physical element in x to the standard element \([ -1, 1 ]\) with coordinate \( \xi \) resulting in the following transformed equation

\[
\frac{\partial u}{\partial t} + \xi \frac{\partial F(u)}{\partial \xi} = 0.
\]

We consider the standard element \( \xi \in [ -1, 1 ] \). The approximation solution can be written as

\[
u_i = \sum_{j=1}^{K} \phi_j(\xi) u_{i,j},
\]

where \( \phi_j \) is the shape function. In this study, DG coefficients are used which means \( \phi_j \) is used as both the shape function and the weighting function.

B. Review of the optimization scheme

For high frequency wave propagation problems, the waves can be severely damped. Based on a Fourier analysis of the CPR method, it has large dissipation and dispersion errors for large wave numbers. The dispersion and the dissipation relations of the 4\textsuperscript{th} order CPR scheme are shown in Figure 1, where \( \Omega \) is the non-dimensional wave number (\( \Omega = \omega \cdot \Delta x \)). For the analysis, an upwind Riemann flux was used.

![Figure 1. The dispersion and the dissipation errors of the 4\textsuperscript{th} order CPR schemes](image-url)
In order to maximize the range of waves that can be resolved accurately, the DRP method \[^{[15,24]}\] was used to optimize the high-order finite difference schemes. For 1D problems, the approximation of the first order spatial derivative \( \frac{\partial u}{\partial x} \) on a uniform grid for a finite difference scheme is given by

\[
\left( \frac{\partial u}{\partial x} \right)_i \approx \frac{1}{\Delta x} \sum_{j=-N}^{M} a_j u(x_i + j\Delta x),
\]

with \( M \) values to the right and \( N \) values to the left of the current point \( i \). Rather than using the Taylor series expansion to determine the coefficients \( a_j \), they are determined by requiring the Fourier transform of the finite difference scheme on the right hand side of Eq. (14) to be a close approximation of the partial derivative on the left hand side. The readers can refer to \[^{[15,24]}\] for more details.

To achieve a similar benefit, in the CPR scheme, the Fourier components of certain frequencies are introduced into the basis functions. For each element, we define the following three spaces: polynomial, Fourier and hybrid

\[
B = \text{span}(1, \xi, \xi^2, \xi^3, \ldots),
\]

\[
B = \text{span}(\sin(a_1 \xi), \cos(a_1 \xi), \sin(a_2 \xi), \cos(a_2 \xi), \ldots),
\]

\[
B = \text{span}(1, \xi, \xi^2, \xi^3, \ldots, \sin(a_1 \xi), \cos(a_1 \xi), \sin(a_2 \xi), \cos(a_2 \xi), \ldots),
\]

where \((a_1, a_2, \ldots)\) are free-parameters. The motivation to use the hybrid space instead of a polynomial space is to obtain a better approximation of broadband wave propagation, because the Fourier terms can exactly represent waves with certain non-dimensional wave numbers, \( \Omega \), and thus yield smaller dispersion and dissipation errors for relatively high frequency waves. At the same time, the monomials are used to achieve a certain order of accuracy with mesh refinement. The free-parameters are optimized to minimize both dispersion and dissipation errors over a specified range of wave numbers.

Because of the Fourier terms in the basis, the exact dispersion relation is satisfied at a certain \( \Omega \). Let’s take a hybrid basis \( B = (1, \xi, \sin(2 \xi), \cos(2 \xi)) \) for instance. Its dispersion and dissipation relations are shown in Figure 2. It is shown that this scheme gives no error at non-dimensional wave number 4 \( (\Omega = \alpha \Delta \xi = 2 \times 2 = 4) \) and less error for wave numbers near 4 or larger than 4, compared with the polynomial basis. Since each element has 4 DOFs, this scheme has no spatial error if a wave has 6.28 DOFs, or the mesh satisfies 6.28 PPW (points per wave).

**Figure 2. Dispersion and dissipation errors of scheme \( B = (1, \xi, \sin(2 \xi), \cos(2 \xi)) \)**

This benefit can be obtained for any free parameter \( \alpha \). Obviously, if we choose a free parameter such that the non-dimensional wave numbers of a problem are all in the region where the FOCPR method has less error than the CPR method, we can obtain more accurate results without increasing the number of DOFs. It should be noted that the 1D coefficients can be used for 2D problems on quadrilateral meshes because the two directions in each element can be treated as if they are decoupled.
III. Numerical tests

To evaluate the performance of the hybrid scheme, several computational aeroacoustic problems were tested using both the CPR and the FOCPR methods. In all the numerical evaluations in this paper, only the hybrid basis \( B = (1, \xi, \sin(\alpha \times \xi), \cos(\alpha \times \xi)) \) was compared with the 4th order CPR method for the sake of simplicity.

A. 1D wave propagation test

The governing equation is given by
\[
\frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} = 0,
\]
with the following initial condition,
\[
u(x,0) = -\sin(\pi x).
\]

An explicit 3rd order Runge-Kutta method was used in time marching. The computational domain is [0, 20]. The \( h \) refinement results are given in Figure 3.

B. 1D Acoustic wave propagation

This 1D problem was computed on a 2D mesh, and the governing equations were the 2D Euler Equations. The computational domain, \([0,20] \times [0,1]\), was initialized with a uniform mean flow from left to right with Mach number \( M = 0.5 \). A perturbation of wave number \( k = 4 \) and wave speed \( \lambda = 1.5 \), \( \epsilon = 1.e^{-7} \) was added from time \( t = 0 \). A uniform quadrilateral mesh was used. The equations were solved with both the CPR and the FOCPR methods in space and an explicit 3rd order Runge-Kutta method in time. The analytic solutions are

\[
p = \bar{p}(1 + \epsilon \cos[k(x - \lambda t)]),
\]
\[
\rho = \bar{\rho}(1 + \epsilon \cos[k(x - \lambda t)]),
\]
\[
u = \bar{u}(1 + \frac{\epsilon}{\gamma M} \cos[k(x - \lambda t)]).
\]
The pressure contour at $t = 20$ is shown in Figure 4. The L2 norm error was calculated after 10 periods. The $h$ refinement results are given in Figure 5.

![Figure 4. Pressure contours and mesh](image)

**Figure 4. Pressure contours and mesh**

![Figure 5. 1D wave propagation L2 Norm Error vs. $h$ refinement](image)

**Figure 5. 1D wave propagation L2 Norm Error vs. $h$ refinement**

According to Figure 5, 2nd order accuracy is achieved with the $h$ refinement for the FOCPR scheme, while 4th order accuracy is achieved for the CPR scheme. We note that the lowest two points of the CPR scheme do not give 4th order accuracy. It is because machine zero has been reached. When the PPW of the problem matches that of the FOCPR scheme, the error from the space discretization diminishes. Similar to the 1D linear wave propagation problem, we should note that the solutions of FOCPR scheme are not as accurate as that of the CPR method on smaller mesh sizes. The results of this numerical test confirm the benefit of the FOCPR method to 2D Euler Equations, which are usually solved in CAA problems.

C. Multi-geometry scattering problem

This case is the Category 2 Problem 1 from the Fourth CAA Workshop. It was the scattering of sound generated by a spatially distributed, axisymmetric, acoustic source from two rigid circular cylinders. The governing equations were the unsteady Euler equations with a time dependent source term in the energy equation.

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = S, \quad (23a)$$

where $Q$ is the vector of conserved variables, $E$ and $F$ are the inviscid flux vectors in x and y directions:

$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho \nabla w \\ u(E + p) \end{bmatrix}, \quad E = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho \nabla w \\ \rho \nabla w^2 + p \\ u(E + p) \end{bmatrix}, \quad F = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho \nabla w \\ \rho \nabla w^2 + p \\ v(E + p) \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ 0 \\ 0 \\ e^{-\ln2(x^2+y^2)^{1/2} + \sin(\omega t)} \end{bmatrix}. \quad (23b)$$

The acoustic source used in this case had a transient term expressed in the following form

$$S = e^{-\frac{1}{2}(x^2+y^2)^{1/2}} \sin(\omega t) f(t), \quad (24)$$

$$f(t) = \min \left(1, \left(\frac{t}{t_0}\right)^3\right). \quad (25)$$

The following parameters were chosen in the present study: $\omega = 8\pi$, $t_0 = 4$. Since the configuration was symmetric, only the upper half of the physical domain was considered. The coarse mesh is shown in Figure 6 and it
has 11359 cells. The resolution of the fine mesh is four times of the coarse mesh in both directions. The entire computational domain was a rectangle with length \(a = 30\), width \(b = 15\). The grid within \(a_1 = 18\), \(b_1 = 9\) was nearly uniform with a resolution of 6.28 points per wave. The mesh was coarsened in the outer region with an expansion factor of 1.1 to minimize the influence of reflection from the outer boundary. An explicit 3rd order SSP Runge-Kutta method was used for time integration. The rms pressure was computed in the last 4 periods after force coefficients became periodic. The computed pressure field at a certain time is shown in Figure 7. The computed rms pressure along the center line is compared with the analytical solution in Figure 8.

Figure 6. Computational grid for the two-cylinder scattering problem

Figure 7. Computed pressure field

Figure 8. Comparison of the computational and analytical RMS pressure along the center line

In this problem, the FOCPR scheme was optimized for waves at \(4\pi = 12.5\) PPW to match the PPW for the cells near the cylinders, because the sound reflected from the cylinders affected the final solution very much. Figure 8 shows that the solution computed with the FOCPR method converges to the analytic solution with mesh refinement. In fact, the solutions computed with both the CPR and the FOCPR methods on the fine mesh agree with the
analytical solution well. On the coarser mesh, the FOCPR method gave more accurate results than the CPR method. The performance of the FOCPR method was better than the CPR method on relatively coarser meshes. This can lead to a smaller number of DOFs for a given error-threshold for CAA problems.

D. Cascade-gust interaction

This is a case from the Fourth CAA Workshop, i.e., Category 3 Problem 2. The two-dimensional geometry and the coarse and fine meshes are shown in Figure 9. The mesh is composed degree 4 elements generated with Gmsh. The coarse mesh has 2044 elements and the fine mesh has 8372 elements. The geometry is the unrolled section of a realistic three-dimensional fan outlet guide vane stator. It has a gap-to-chord ratio of $d/c = 2/3$ with the inflow and outflow planes located at $x_{\pm} = \pm \frac{3}{2} c$. The time-averaged inflow/outflow conditions are:

\[
\text{Inflow conditions: } \begin{cases}
P_i &= 1 \\
\bar{T}_i &= 1 \\
\vec{a}_i &= 36^\circ
\end{cases}, \quad \text{outflow condition: } \frac{\bar{P}_o}{P_i} = 0.92,
\]

where $\bar{P}_i$ and $\bar{T}_i$ are the normalized inflow plane mean stagnation pressure and mean stagnation temperature. $\vec{a}_i$ is the mean flow angle and $\bar{P}_o$ the normalized outflow plane mean static pressure. The flow is assumed to be inviscid and isentropic throughout the domain.

The inflow gust (produced, say, by the wake of an upstream blade row) is given, at the inflow plane, by

\[
\vec{u}_y(y,t) = \left\{ a_1 \cos(k_y y - \omega t) + a_2 \cos(2(k_y y - \omega t)) + a_3 \cos(3(k_y y - \omega t)) \right\} \hat{e}_y, \quad \omega = \frac{3 \pi}{4}, \quad k_y = \frac{11 \pi}{9},
\]

\[
\hat{e}_\beta = \cos(\beta) \hat{e}_x - \sin(\beta) \hat{e}_y, \quad \beta = 44^\circ,
\]

\[
\begin{align*}
a_1 &= 5 \times 10^{-3} \\
a_2 &= 3 \times 10^{-3} \\
a_3 &= 7 \times 10^{-4}
\end{align*}
\]

where $\omega$ is the fundamental reduced frequency, $k_y$ is the transverse wavenumber, and the $a_i$’s are the gust harmonic amplitudes.

The steady solution was first obtained for the specified boundary conditions and then the three different frequencies of the inflow gust were added at the inflow plane, separately.

In the present simulation, a single passage was considered in the calculation for both the steady and unsteady situations in order to achieve the highest frequency. For the steady case, the flow was considered periodic at every passage, while for the unsteady perturbation case, a constant phase difference between adjacent blades was assumed. A brief description of the treatment of the phase difference and phase-lagged boundary condition is given next. For a detailed explanation, readers can refer to [13].

The application of the phase-lagged boundary condition requires first storing the time variation of the fluid properties at the passage boundaries. They are then used to update the fluid properties associated with the other blades, which shifted in time with the phase of blade motion. Let us take the first component of the gust as an example. The width of the single passage is $\Delta y$, thus the phase difference $\theta$ between adjacent blades is $k_y \Delta y$, which corresponds to a shift of $\frac{k_y \Delta y}{\omega}$ in time. Let us define the interior solution at the boundary above the blade at any time to be $f_A(t)$, below the blade $f_B(t)$, and define the exterior boundary (ghost) condition at the boundary above the blade at any time to be $f_C(t)$, below the blade $f_D(t)$, respectively. $f_A(t)$ and $f_B(t)$ can be solved and stored. Then the exterior boundary (ghost) conditions can be obtained by

\[
f_C(t) = f_B \left( t - \frac{k_y \Delta y}{\omega} \right),
\]

\[
f_D(t) = f_A \left( t - \frac{2\pi - k_y \Delta y}{\omega} \right),
\]

which are the prior solutions at the boundaries. It requires that $\frac{k_y \Delta y}{\omega}$ divided by the time step should be an integer. For the initial steps of computation, no prior information is available. During these steps, the boundaries are treated as being periodic. The errors introduced by this treatment increase the number of oscillations required for convergence.

Both the CPR and FOCPR methods were used to compute the steady solution on the coarse and fine meshes. The free parameter of the FOCPR method here was chosen to be 2.0, which corresponds to 6.28 PPW. The steady pressure contours are shown in Figure 10.
For the unsteady perturbation, periodicity was achieved for all the simulations with different schemes. Figure 11 shows the history of the drag coefficient for the simulation with $3\omega$ frequency computed with the 4th order CPR method on the coarse mesh. The axial velocity perturbation fields are shown in Figure 12. Table 1 shows the predicted amplitude of the unsteady blade surface pressure at three selected locations for each of the 3 frequencies computed with the FOCPR method on the coarse mesh. Table 2 shows the results computed with the 4th order CPR method on the coarse mesh. Table 3 shows the results computed with the 6th order CPR method on the coarse mesh and Table 4 shows the results computed with 4th order CPR on the fine mesh. The results were given as SPL(dB)

$$SPL = 20 \log \left( \frac{p_{rms}}{p_{ref}} \right),$$

where $p_{ref} = 20\mu Pa$, $p_{rms}$ was computed for 4 periods after periodicity was achieved. To demonstrate mesh and order independent solution convergence, the 4th and 6th order CPR schemes were used for the coarse mesh while the 4th order CPR scheme was used on the fine mesh. The RMS pressures computed with the 6th order CPR scheme on the coarse mesh differ less than 1% from those computed with the 4th order CPR scheme on the fine mesh for all frequencies at all recorded locations, as shown in Table 3, 4. Therefore, the fine mesh results are used as the “true solution”. The performance of the CPR and the FOCPR schemes are compared in Figure 13. The SPL(dB) values at $x = 0.25$ on the vane suction side for the gust component at frequency of $3\omega$ were compared, because the highest frequency perturbation presented the most severe challenge for numerical methods.

![Figure 9. Computational grids for the cascade-gust interaction problem](image)

![Figure 10. Pressure distribution for a steady flow over a cascade(pressure normalized by the inflow mean stagnation pressure)](image)
Figure 11. The history of drag coefficient

Figure 12. The axial velocity perturbation fields for (a) frequency $\omega$ (b) frequency $2\omega$ (c) frequency $3\omega$ using the FOCPR method on the coarse mesh
Table 1. Acoustic pressure spectrum on the vane using the FOCPR method on the coarse mesh

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Suction Side SPL(dB)</th>
<th>Pressure Side SPL(dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>x/c=-0.25</td>
<td>x/c=0.00</td>
<td>x/c=0.25</td>
</tr>
<tr>
<td>ω</td>
<td>141.897</td>
<td>142.166</td>
</tr>
<tr>
<td>2ω</td>
<td>132.019</td>
<td>118.348</td>
</tr>
<tr>
<td>3ω</td>
<td>110.349</td>
<td>98.6303</td>
</tr>
<tr>
<td>x/c=0.00</td>
<td>141.763</td>
<td>142.508</td>
</tr>
<tr>
<td>x/c=0.25</td>
<td>129.662</td>
<td>127.595</td>
</tr>
<tr>
<td>x/c=0.25</td>
<td>116.984</td>
<td>112.712</td>
</tr>
</tbody>
</table>

Table 2. Acoustic pressure spectrum on the vane using the 4th order CPR method on the coarse mesh

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Suction Side SPL(dB)</th>
<th>Pressure Side SPL(dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>x/c=-0.25</td>
<td>x/c=0.00</td>
<td>x/c=0.25</td>
</tr>
<tr>
<td>ω</td>
<td>141.763</td>
<td>141.661</td>
</tr>
<tr>
<td>2ω</td>
<td>131.946</td>
<td>119.081</td>
</tr>
<tr>
<td>3ω</td>
<td>110.286</td>
<td>101.4</td>
</tr>
<tr>
<td>x/c=0.00</td>
<td>141.661</td>
<td>126.209</td>
</tr>
<tr>
<td>x/c=0.25</td>
<td>129.761</td>
<td>115.896</td>
</tr>
<tr>
<td>x/c=0.25</td>
<td>116.864</td>
<td>112.694</td>
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</tbody>
</table>

Table 3. Acoustic pressure spectrum on the vane using the 6th order CPR method on the coarse mesh

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Suction Side SPL(dB)</th>
<th>Pressure Side SPL(dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>x/c=-0.25</td>
<td>x/c=0.00</td>
<td>x/c=0.25</td>
</tr>
<tr>
<td>ω</td>
<td>142.165</td>
<td>142.214</td>
</tr>
<tr>
<td>2ω</td>
<td>132.076</td>
<td>118.035</td>
</tr>
<tr>
<td>3ω</td>
<td>111.488</td>
<td>95.968</td>
</tr>
<tr>
<td>x/c=0.00</td>
<td>142.214</td>
<td>126.415</td>
</tr>
<tr>
<td>x/c=0.25</td>
<td>129.456</td>
<td>115.586</td>
</tr>
<tr>
<td>x/c=0.25</td>
<td>117.81</td>
<td>108.451</td>
</tr>
</tbody>
</table>

Table 4. Acoustic pressure spectrum on the vane using the 4th order CPR method on the fine mesh

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Suction Side SPL(dB)</th>
<th>Pressure Side SPL(dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>x/c=-0.25</td>
<td>x/c=0.00</td>
<td>x/c=0.25</td>
</tr>
<tr>
<td>ω</td>
<td>142.203</td>
<td>142.225</td>
</tr>
<tr>
<td>2ω</td>
<td>132.096</td>
<td>117.803</td>
</tr>
<tr>
<td>3ω</td>
<td>111.904</td>
<td>96.493</td>
</tr>
<tr>
<td>x/c=0.00</td>
<td>142.167</td>
<td>126.547</td>
</tr>
<tr>
<td>x/c=0.25</td>
<td>129.432</td>
<td>115.848</td>
</tr>
<tr>
<td>x/c=0.25</td>
<td>118.286</td>
<td>113.281</td>
</tr>
</tbody>
</table>

Figure 13. Acoustic pressure spectrum on the vane at x = 0.25, suction side vs. 1/sqrt(nDOFs) for frequency of 3ω

It is shown that the SPLs computed with the FOCPR method on the coarse mesh are closer to the fine grid results than those computed with CPR method, demonstrating FOCPR’s higher accuracy when the mesh barely has enough resolution to properly resolve high frequency waves. Note that on the fine mesh, the SPLs computed with both the CPR and FOCPR schemes are very similar, indicating convergence independent of the numerical method. From Figure 9, it is obvious that the elements are very small at the leading edge and the trailing edge to satisfy the geometry resolution requirement. But the elements at the inlet, outlet, top and bottom boundaries are much larger to reduce the number of DOFs. The non-dimensional wave numbers vary significantly from one region of the mesh to
another. Some of the wave numbers are located outside the region where the FOCPR method has smaller error than the CPR method. As a result, the benefit from the FOCPR scheme was not as obvious.

IV. Conclusions

The FOCPR method was developed to improve the resolution of the CPR formulation for broadband waves. In this paper, the FOCPR method was evaluated with several numerical tests. In the numerical evaluation of the 1D linear wave propagation, the FOCPR method was able to exactly represent the wave whose non-dimensional wavenumber matched the scheme’s non-dimensional wavenumber and achieved 2nd order accuracy with mesh refinement. In the 1D wave propagation problem using the 2D Euler equations, the FOCPR method also showed similar benefits. For both benchmark problems from the 4th Computational Aeroacoustic(CAA) Workshop (Category 2 Problem 1, Category 3 Problem 2) when the mesh resolution is barely enough to resolve the high-frequency components, the FOCPR method shows a clear advantage in accuracy on the coarse mesh. Once the mesh was refined, the benefit was not as obvious because in the limit of diminishing mesh size, the CPR scheme is always more accurate by design. As a result, for problems with widely varying non-dimensional wave numbers, the advantage of the FOCPR method may be limited.

Acknowledgements

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