An Efficient Spectral (Finite) Volume Method for Conservation Laws On Unstructured Grids

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ABSTRACT: A conservative, efficient high-order finite volume method named Spectral Volume (SV) method has been developed for conservation laws on triangular grids. Each triangular cell is further subdivided into subcells, and mean state variables at the subcells are used to reconstruct a high-order polynomial in the triangular cell. A Godunov-type finite volume method is then used to generate high-order updates for the subcell means. In addition, TVD (Total Variation Diminishing) and TVB (Total Variation Bounded) limiters are introduced in the SV method to remove spurious oscillations near discontinuities. A very desirable feature of the SV method is that the reconstruction is identical for cells of the same type with similar partitions, and that the reconstruction stencil is always non-singular, in contrast to the memory and CPU-intensive reconstruction in a high-order k-exact finite volume (FV) method. The high-order accuracy of the SV method is demonstrated for the Burger’s equation with and without discontinuities.

KEY WORDS: Conservation Laws, Unstructured Grid, Spectral Volume, High-Order
1. Introduction

Significant progresses have been made in the last three decades on numerical algorithms for conservation laws, which govern many physical disciplines such as fluid dynamics, electromagnetics, aeroacoustics, to name just a few. It is very desirable that these algorithms satisfy all of the following requirements: 1) conservative, 2) geometrically flexible, 3) high-order accurate and 4) efficient. Although second-order methods based on structured grids have been used successfully in computational fluid dynamics (CFD) to solve flow over complex configurations, generating smooth computational grids suitable for high-order algorithms is a daunting task. We therefore focus our attention exclusively on unstructured grid-based high-order algorithms, which offer true geometric flexibility.

One of the most successful algorithms for conservation laws is the Godunov method [GOD 59], which laid a solid foundation for the development of modern upwind methods including MUSCL [VAN 79], ENO [HAR 87] and weighted ENO (WENO) schemes [FRI 98, HU 99]. The basic ideas of the Godunov-type method has been successfully extended to unstructured grids, leading to the development of high-order k-exact finite volume method by Barth and Frederickson in [BAR 90], an ENO scheme for unstructured grid by Abgrall in [ABG 94], and WENO schemes by Friedrich [FRI 98], Hu and Shu in [HU 99]. Although high-order accurate finite volume schemes can be obtained theoretically for an unstructured grid by using high-order polynomial data reconstructions, higher than linear reconstructions are rarely used in three dimensions in practice. This is mainly because of the difficulty in finding valid (non-singular) stencils, and the enormous memory required to store the coefficients used in the reconstruction. For each control volume, the reconstruction stencil is unique for an unstructured grid as shown in Figure 1a. A data reconstruction must be performed at each iteration for each control volume. This reconstruction step is the most memory and time consuming in higher than second-order schemes. In a recent implementation of a third-order FV scheme with a quadratic reconstruction in three dimensions by Delanaye and Liu [DEL 99], the average size of the reconstruction stencils is about 50-70. Still there are many singular reconstruction stencils. The size of the reconstruction stencils usually increases non-linearly with the order of accuracy. For a fourth order FV scheme, the average stencil size is estimated to be at least 120. It is very memory and CPU intensive to perform the reconstruction.

More recently, another high-order conservative algorithm called the Discontinuous Galerkin (DG) method was developed by Cockburn, Shu, et al and others [COC 89, 90]. In the DG method, a high-order data distribution is assumed for each element. As a result, the state variable is usually not continuous across element boundaries. The fluxes through the element boundaries are computed using an approximate Riemann solver [ROE 81], similar to a FV method. In fact, the first-order DG and FV methods are identical. Due to the use of Riemann fluxes cross element boundaries, the DG method is fully conservative. A disadvantage of the DG method is that very high-order surface and volume integrals are necessary, which can be expensive to compute. Another high-order conservative scheme for unstructured quadrilateral grids is the multi-domain spectral method on staggered grids developed by Kopriva and Kolias [KOP 96]. The multi-domain spectral method is similar to the spectral element method by Patera [PAT 84], which is not conservative. Although very high-order of accuracy was achievable with both methods, the methods are difficult to extend to other cell types such as triangles, or tetrahedra.

In this paper, we present a new conservative high-order SV method for conservation laws on unstructured grids. Details follow.

2. Spectral (Finite) Volume Method

The idea of the Spectral Volume method is best presented in two dimensions (2D). Its degeneration into 1D and extension to 3D are obvious. Consider a mesh of unstructured triangular cells. Each cell is called a Spectral Volume, denoted by $S_i$, which is further partitioned into subcells named Control Volumes (CVs), indicated by $C_{ij}$, as shown in Figure 1b. To represent the solution as a polynomial of degree $m$ in 2D, we need $N = (m+1)(m+2)/2$ pieces of independent information, or degrees of freedom (DOFs). The DOFs in a SV method are the volume-averaged mean variables at the $N$ CVs. For example, to build a quadratic reconstruction in 2D, we need at least $(2+1)(3+1)/2 = 6$ DOFs. There are numerous ways of partitioning a SV, and not every partition is admissible in the sense that the partition may not be capable of producing a degree $m$ polynomial. Once $N$ mean solutions in the CVs of an admissible SV are given, a unique polynomial reconstruction can be obtained from
Figure 1. (a) A possible reconstruction stencil for a quadratic reconstruction in a high-order finite volume scheme; (b) The partition of a Spectral Volume into six Control Volumes supporting quadratic reconstruction.

\[ p_i(x, y) = \sum_{j=1}^{N} L_j(x, y)\bar{u}_{i,j}, \quad (2.1) \]

where \( p_i(x, y) \in P_m \) (the space of polynomials of degree \( m \) or less), \( L_j(x, y) \in P_m \), \( j = 1, \ldots, N \) are the “shape” functions satisfying

\[ \int_{C_{i,j}} L_q(x, y)dV = V_{i,j}\delta_{q,k}, \quad (2.2) \]

where \( V_{i,j} \) is the volume of \( C_{i,j} \). This high-order polynomial reconstruction facilitates a high-order update for the mean solution of each CV. Consider the following hyperbolic conservation law

\[ u_t + \nabla \cdot F = 0, \quad (2.3) \]

where \( F \) is the flux vector. Integrating (2.3) in each CV, we obtain

\[ \frac{d\bar{u}_{i,j}}{dt} + \sum_{r=1}^{K} (F \cdot n)dA = 0, \quad (2.4) \]

where \( K \) is the total number of faces in \( C_{i,j} \), and \( \bar{u}_{i,j} \) is the volume-averaged solution at \( C_{i,j} \). The flux integral in (2.4) is then replaced by a Gauss-quadrature formula which is exact for polynomials of degree \( m \)

\[ \int_{A_r} (F \cdot n)dA = \sum_{q=1}^{J} w_{rq} F(u(x_{rq}, y_{rq})) \bullet n_r A_r, \quad (2.5) \]

where \( J \) is the number of quadrature points on the \( r \)-th face, \( w_{rq} \) are the Gauss quadrature weights, \((x_{rq}, y_{rq})\) are the Gauss quadrature points. Since the reconstructed polynomials are piece-wise continuous, the solution is usually discontinuous across the boundaries of a SV, although it is continuous across interior CV faces. The fluxes at the interior faces can be computed directly based on the reconstructed solutions at the quadrature points. The fluxes at the boundary faces of a SV are computed using approximate Riemann solvers given the left and right reconstructed solutions, i.e.,

\[ F(u(x_{rq}, y_{rq})) \bullet n_r = F_{Riem}(p_{i,r}(x_{rq}, y_{rq}), p_{i,r}(x_{rq}, y_{rq}), n_r) = -F_{Riem}(p_{i,r}(x_{rq}, y_{rq}), p_{i,r}(x_{rq}, y_{rq}), -n_r), \quad (2.6) \]

where \( p_{i,r}(x_{rq}, y_{rq}) \) is the reconstructed polynomial in a neighboring SV sharing face \( r \) with the SV in consideration, \( S_i \). Obviously, the approximate Riemann solver must satisfy

\[ F_{Riem}(p_{i,r}(x_{rq}, y_{rq}), p_{i,r}(x_{rq}, y_{rq}), n_r) = -F_{Riem}(p_{i,r}(x_{rq}, y_{rq}), p_{i,r}(x_{rq}, y_{rq}), -n_r), \]

to achieve discrete conservation. It has been shown [WAN 02] that this SV scheme is \((m+1)\)-th order accurate. In addition, the scheme is compact in the sense that a high-order polynomial is reconstructed in each SV without using any data from neighboring SVs. This property can potentially translate into significantly reduced communication cost compared to a k-exact FV scheme (for example) when implemented on parallel computers.

Note that one of the subtle differences between a FV method and a SV method is that all the CVs in a SV use the same data reconstruction. As a result, it is not necessary to use a Riemann flux or flux splitting for the interior boundaries between the CVs inside a particular SV because the state variable is continuous across the interior CV boundaries. Riemann fluxes are only necessary at the boundaries of the SV.

For time integration, we use TVD Runge-Kutta schemes from [SHU 88]. The SV method shares many advantages with the DG method [COC 89, 90] in that it is compact which is suitable for parallel computing, high-order accurate, conservative, and capable of handling complex geometries. The SV method is expected to be more...
efficient than the DG method because high-order volume integrals are avoided, and lower order surface Gauss integral formula can be used \((m\text{-th order vs. } 2m\text{-th order})\).

3. Convergence of the SV method and Data Limiting

Given a non-polynomial continuous function in a certain domain, the Lagrange interpolation polynomial based on given nodal values of the function on equidistant nodes may diverge from the original function. This was shown by Runge using the function \(1/(1+x^2)\) on domain \([-5, 5]\). In fact, this "Runge phenomenon" has also been observed in the data reconstruction using means on equidistant CVs. It was shown [WAN 02] that a fourth-order SV scheme using equidistant CVs is not convergent with grid refinement. The errors in the computational results actually increase with grid refinement, showing a diverging behavior. On the other hand, SV partitions using Gauss-Lobatto points produced grid-convergent results. A general observation is that CVs need to cluster near the boundaries of a SV to produce convergent SV schemes. This observation is also true in 2D [WAN 02]. For example, the more uniform partition shown in Figure 2d is not convergent, while the non-uniform partition in Figure 2e is convergent. It is established that the Lebesgue constant for a partition defined as [CHE 95, WAN 02]

\[
\|\Gamma_n\| = \max_{(x,y) \in \Gamma} \sum_{j=1}^{N} \left| L_j(x,y) \right|
\]  

is a good quality measure of the partition because

\[
\|u - p_i(x,y)\| \leq (1 + \|\Gamma_n\|) \|u - p_i^\ast\|
\]

where \(u\) is any continuous function, and \(p_i^\ast\) is the best uniform approximation to \(u\) on \(E\) (the standard equilateral triangle). Therefore, the smaller the Lebesgue constant, the better the interpolation is to be expected.

For solutions containing discontinuities, it is necessary to perform data limiting to maintain solution monotonicity and sometimes stability. A limiter in a high-order numerical method such as the SV method should satisfy the following two requirements: 1) non-oscillatory, sharp resolution of discontinuities, and 2) recover the full formal order of accuracy away from the discontinuities. To that end, a TVB limiter [SHU 87] has been implemented. Denote

\[
\Delta u_{rq} = p_r(x_q, y_{rq}) - \overline{u}_{i,j}, \quad r = 1, \cdots, K; \quad q = 1, \cdots, J.
\]

Following the TVB idea, if

\[
\Delta u_{rq} \leq 4M h_{rq}^2, \quad r = 1, \cdots, K; \quad q = 1, \cdots, J,
\]

it is not necessary to do any data limiting. In (3.3), \(M\) represents some measure of the second derivative of the solution, and \(h_{rq}\) is the distance from point \((x_{rq}, y_{rq})\) to the centroid of \(C_{i,j}\). If for any value of \(r\) and \(q\), (3.3) is violated, it is assumed that \(C_{i,j}\) is near a steep gradient and data limiting is necessary. Instead of using the polynomial \(p_r(x, y)\) in \(C_{i,j}\), we assume linear data distribution in \(C_{i,j}\), i.e.,

\[
u_{i,j}(x, y) = \overline{u}_{i,j} + \nabla u_{i,j} \cdot (r - r_{i,j}), \quad \forall r \in C_{i,j},
\]

where \(r_{i,j}\) is the position vector of the centroid of \(C_{i,j}\). In order to achieve the highest resolution, we need to maximize the magnitude of the solution gradient \(\nabla u_{i,j}\). At the same time, we require that the reconstructed solutions at the quadrature points of \(C_{i,j}\) satisfy the following monotonicity constraint:

\[
\overline{u}_{i,j}^\text{min} \leq u_{i,j}(x_{rq}, y_{rq}) \leq \overline{u}_{i,j}^\text{max},
\]

where \(\overline{u}_{i,j}^\text{min}\) and \(\overline{u}_{i,j}^\text{max}\) are the minimum and maximum cell-averaged solutions among all its neighboring CVs sharing a face with \(C_{i,j}\). A very efficient approach can be used to compute the gradient. In this approach, we avoid a separate data reconstruction by reusing the polynomial reconstruction already available for the SV. For each CV, we use the gradient of the reconstructed polynomial at the CV centroid, i.e.,

\[
\nabla u_{i,j} = \left[ \frac{\partial p_i}{\partial x}, \frac{\partial p_i}{\partial y} \right]_{r_{i,j}}.
\]

This gradient is then limited if necessary to satisfy (3.5). If any of the reconstructed variable at the quadrature points is out of the range \([\overline{u}_{i,j}^\text{min}, \overline{u}_{i,j}^\text{max}]\), the gradient is limited, i.e.,

\[
\nabla u_{i,j} \leftarrow \varphi \nabla \overline{u}_{i,j},
\]

where \(\varphi\) is a saturation function.
where \( \varphi \in [0, 1] \). Note that if parameter \( M = 0 \), the TVB limiter is TVD (total variation diminishing), which strictly enforces monotonicity by sacrificing accuracy near extrema.

### 4. Analysis of the Second-Order SV Scheme in One Dimension

In order to analyze the characteristics of the SV method, we consider the following one-dimensional scalar conservation laws

\[
\frac{\partial u(x,t)}{\partial t} + c \frac{\partial u(x,t)}{\partial x} = 0, \quad \text{in } [a, b] \times [0, T] \tag{4.1a}
\]

\[
u(x,0) = u_0(x), \quad \forall x \in [a, b] \tag{4.1b}
\]

where \( c > 0 \) is the wave speed. We further assume that the computational domain is discretized into \( N \) equidistant spectral volumes, and each spectral volume is partitioned two equidistant control volumes:

\[
h_i = (b - a)/N, \quad i = 1, \ldots, N
\]

\[
\Delta = h_{i,j} = h_i/2 = (b-a)/2N, \quad i = 1, \ldots, N \quad j = 1, \ldots, 2
\]

For a second order SV scheme, a linear reconstruction is used in each SV. Therefore it is obvious that the second-order SV scheme can be written as

\[
\frac{d\bar{u}_{i,1}}{dt} = -\frac{c}{\Delta} \left( \bar{u}_{i,1} + \bar{u}_{i,2} - 3\bar{u}_{i-1,2} + \bar{u}_{i-1,1} \right), \quad \text{(4.2a)}
\]

\[
\frac{d\bar{u}_{i,2}}{dt} = -\frac{c}{\Delta} \left( \bar{u}_{i,2} - \bar{u}_{i,1} \right), \quad \text{(4.2b)}
\]

where \( \bar{u}_{i,j} \) represents the volume-averaged mean in the \( j \)-th CV of the \( i \)-th SV. Note that the cell-averaged value on a CV can be approximated by the value at the cell-center with a second-order accuracy. Therefore (4.2) can be expressed as node-based finite-difference schemes without loss of accuracy:

\[
\frac{d\bar{u}_{i,1}}{dt} = -\frac{c}{\Delta} \left( \frac{u_{i+1} + u_{i} - 3u_{i-1,2} + u_{i-1,1}}{2} \right) \quad \text{(4.3a)}
\]

\[
\frac{d\bar{u}_{i,2}}{dt} = -\frac{c}{\Delta} \left( u_{i,2} - u_{i,1} \right) \quad \text{(4.3b)}
\]

It is interesting that while the overall SV scheme is second-order accuracy in space, neither of the two schemes (4.3a) and (4.3b) is second-order accurate if used alone because it is obvious that (4.3b) is the well-known first-order upwind scheme! Therefore, it seems that some sort of error canceling is happening here. We can easily derive the modified equations for (4.3a) and (4.3b) to be

\[
\frac{\partial \bar{u}_{i,1}}{\partial t} + c \frac{\partial \bar{u}_{i,1}}{\partial x} = -\frac{\Delta}{2} \frac{\partial^2 \bar{u}_{i,1}}{\partial x^2} + O(\Delta^2) \quad \text{(4.4a)}
\]

\[
\frac{\partial \bar{u}_{i,2}}{\partial t} + c \frac{\partial \bar{u}_{i,2}}{\partial x} = +\frac{\Delta}{2} \frac{\partial^2 \bar{u}_{i,2}}{\partial x^2} + O(\Delta^2) \quad \text{(4.4b)}
\]

Note that scheme (4.3a) has a first-order negative artificial dissipation term if used throughout the computational domain, while (4.3b) has a positive first-order dissipation term. Scheme (4.3a) is not only first-order accurate, but also unconditionally unstable if used alone! Adding (4.3a) and (4.3b) together, we obtain

\[
\frac{d(u_{i+1} + u_{i})}{dt} = -\frac{c}{\Delta} \left( -u_{i+1} + 3u_{i,2} - 3u_{i-1,2} + u_{i-1,1} \right) \quad \text{(4.5)}
\]

Expanding (4.5) at the enter of the \( i \)-th SV \( S_i \), we obtain:

\[
\frac{\partial \bar{u}_{i}}{\partial t} + c \frac{\partial \bar{u}_{i}}{\partial x} = O(\Delta^2) \quad \text{(4.6)}
\]

where \( \bar{u}_i \) represents the state variable at the center of \( S_i \). That is to say that the cell-averaged state variable in the spectral volume \( S_i \) does satisfy the second-order accuracy requirement.

It is clear that the semi-discrete second-order FSV scheme is consistent. However, the stability of the FSV scheme is not obvious. In order to analyze the stability condition, we discretize the time derivative with the second-order TVD Runge-Kutta scheme. Using a Fourier analysis, we can prove that the scheme is stable when
\[ \sigma = \frac{c\Delta t}{\Delta} \leq 1. \]  \hfill (4.7)

This stability condition has also been numerically verified. This condition compares favorably with the stability condition of the second-order DG scheme, which is

\[ \sigma' = \frac{c\Delta t}{2\Delta} \leq 1/3, \quad \text{i.e.,} \quad \sigma \leq 2/3 \]  \hfill (4.8)

5. Accuracy Study with 2D Burgers Equation

In this case, we test the accuracy of the SV method on the two-dimensional non-linear wave equation:

\[ \frac{\partial u}{\partial t} + \frac{\partial u^2}{2\partial x} + \frac{\partial u^2}{2\partial y} = 0, \quad -1 \leq x \leq 1, -1 \leq y \leq 1, \]

\[ u(x, y, 0) = \frac{1}{4} + \frac{1}{2} \sin \pi (x + y), \quad \text{periodic b.c.} \]

The initial solution is smooth. Due to the non-linearity of the Burgers equation, discontinuities will develop in the solution. Therefore we also test the capability of the method to achieve uniform high-order accuracy away from discontinuities. At \( t = 0.1 \), the exact solution is still smooth. The numerical simulation is therefore carried out until \( t = 0.1 \) without the use of limiters on the irregular grid as shown in Figure 3. In Figure 4, the \( L_1 \) and \( L_\infty \) errors on the irregular grid are displayed. Generally speaking, the SV schemes achieved the designed order of accuracy except with the Type 1 linear SV. The Type 1 linear SV has difficulty in achieving second-order accuracy on the irregular grid in both norms.

Figure 2. Possible Partitions of a Triangular Spectral Volume

Figure 3. An Unstructured "10x10x2" Grid
At $t = 0.45$, the exact solution has developed two shock waves as shown in Figure 5. A limiter is necessary to handle the discontinuities. Shown in Figure 5 are the exact solution and the computed numerical solutions with the Type 2 quadratic SV on the 40x40x2 irregular grid using the TVD limiter, i.e., $M$ was taken to be 0. Note that the limiter produced a very good solution.

![Figure 4](image)

Figure 4. Computed $L_1$ and $L_\infty$ Errors with Different SV Schemes at $t = 0.1$

In order to estimate the numerical order of accuracy for the solution away from the discontinuities at $t = 0.45$, $L_1$ and $L_\infty$ errors in the smooth region $[-0.2, 0.4] \times [-0.2, 0.4]$ are computed. Without the use of the limiter, the solution quickly diverged after shock waves were developed in the solution. The parameter $M$ was set to be 400 in the computation. The $L_1$ and $L_\infty$ errors with the best performing SVs for a given order of accuracy are presented in Figure 6. Obviously, with this choice of $M$, the designed order of accuracy was achieved away from discontinuities.

6. Conclusions

A high-order Spectral (Finite) Volume method has been developed for conservation laws on unstructured triangular meshes. Each mesh cell forms a spectral volume, and the spectral volume is further partitioned into polygonal control volumes. High order schemes are then built based on the CV-averaged solutions. A unique feature of the SV method is that a universal reconstruction can be obtained if all spectral volumes are partitioned in a similar manner. However, the way in which a SV is partitioned into CVs affects the convergence property of the resultant numerical scheme. A criterion based on the Lebesgue constant has been developed and used successfully to determine the quality of various partitions. Symmetric, stable, and convergent linear, quadratic and cubic SVs have been obtained, their performance assessed on a model test problem.

An accuracy study with 2D scalar conservation laws has been carried out, and the order of accuracy claim has been numerically verified for convergent SVs. TVD and TVB limiters have been developed for non-oscillatory capturing of discontinuities, and found to perform well. The TVB limiters with a properly selected parameter ($M$) are capable of maintaining uniformly high-order accuracy away from discontinuities.

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References


