Reduced equivalent form of a financial structure

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Abstract

In the two-date model of a financial exchange economy considered in Aouani and Cornet (2009), we show that, if agents’ portfolio restrictions are represented by linear inequality constraints or satisfy Hart (1974)’s Weak No Market Arbitrage condition, then the financial structure is equivalent in terms of consumption equilibria to its reduced form. Building upon the equilibrium existence result for reduced financial economies (Aouani and Cornet, 2009), existence of financial equilibrium is shown under standard assumptions on the consumption side and under the aforementioned assumption on the financial side.

Keywords: Restricted participation, financial exchange economy, reduced financial structure, equivalent financial structure, arbitrage-free prices, consumption equilibrium.

JEL: C62, D52, D53

1. Introduction

Restricted participation to financial markets refers to the fact that agents face exogenous constraints on their portfolio holdings. These constraints are usually described by a collection of subsets \((Z_i)\), one for each agent \(i \in I\) of the portfolio space \(\mathbb{R}^J\). The economic relevance and interest in considering restricted participation dates back to the seminal papers of Radner (1972) where agents face short sales constraints and Siconolfi (1989), Cass (1984, 2006) for general closed convex portfolio sets \(Z_i\). It is also well known that the presence of such portfolio constraints is a natural cause of market incompleteness – even if there exist enough assets to hedge all risks – and allows to capture a wide range of imperfections in the financial markets, such as collateral requirements, margin requirements, “combo” sales, short selling constraints, and more generally institutional constraints. We refer the reader to Elsinger and Summer (2001) who provide an extensive discussion of these institutional constraints and how to model them in a general financial framework.

The equilibrium existence problem in the context of restricted participation had a renewed interest since the first work by Siconolfi (1989), and Cass (1984, 2006). Linear equality constraints...
are considered by Balasko et al. (1990) with nominal assets, and by Polemarchakis and Siconolfi (1997) with real assets, whereas Aouani and Cornet (2009) study linear equality and inequality constraints with either nominal or numéraire assets. More recently, the “general” case of portfolio sets which are closed and convex subsets of the space of all possible portfolios, as in Siconolfi (1989), is considered by Angeloni and Cornet (2006) and Aouani and Cornet (2009) for real assets, and by Martins-da-Rocha and Triki (2005), Hahn and Won (2007), and Cornet and Gopalan (2010) when assets are nominal.

A key step in the proof of existence of a financial equilibrium with nominal, numéraire or real assets is to show that equilibrium portfolios can be, a priori, chosen in a bounded set. This is a standard argument if there are no redundant assets, or equivalently if the payoff matrix has full column rank. With unrestricted participation, there is no loss of generality in making this “Full Rank Assumption”. Indeed, one can easily show that a situation in which redundant assets are available can be converted into one in which they are not by removing the redundant assets (which practically amounts to deleting redundant columns from the payoff matrix). Matters are much more complicated when agents’ participation to financial markets is constrained. In fact, as emphasized by Balasko et al. (1990), one significant source of restricted participation is financial intermediation which typically involves redundancy. So there are no a priori grounds for the standard Full Rank Assumption in the presence of restricted participation. This fact constitutes a major obstacle to the application of fixed point theorems which are usually used to show existence of equilibrium.

With portfolio sets defined by linear equality constraints, Balasko et al. (1990) develop a procedure to overcome this obstacle. They show how to transform agents’ financial opportunities to obtain a financial economy in which each agent’s portfolio choice set is a subspace having the same dimension as the wealth space it generates; a non-redundancy-type condition used by Siconolfi (1989) to show existence of equilibrium. Moreover, every equilibrium in the transformed economy leads to an equilibrium in the original one (of course, the transformation would be of no avail if the latter result did not hold). Aouani and Cornet (2009) extend the above analysis to portfolio sets defined by linear equality and/or inequality constraints. By appropriately modifying agents’ portfolio sets, they obtain a new – say reduced – financial structure satisfying a non-redundancy-type condition weaker than the one in Siconolfi (1989), keeping the correspondence between the equilibria. Furthermore, we show existence of equilibrium for reduced financial economies.

The main purpose of this paper is to go beyond the case of linear constraints. We provide an existence result when the financial structure satisfies Hart (1974)’s weak no market arbitrage condition (WNMA in the text). Although existence of equilibrium was the driving force of this work, the proof of existence of equilibria will be merely an immediate consequence of the conjunction of our main result concerning existence of reduced equivalent financial structures and the equilibrium existence result for reduced financial economies in Aouani and Cornet (2009). This approach allows generalizing all previous existence results for nominal or numéraire assets, and represents the first contribution of this paper to the literature. The key contribution of this paper is to provide a novel approach to the problem posed by dealing with redundant assets. Since simply removing redundant assets in the presence of portfolio restrictions would considerably change the nature of the market by altering wealth transfer sets, we propose instead, to remove some of the
redundant portfolios. In Aouani and Cornet (2008), these portfolios are labeled as useless and a justification for this term is provided. More precisely, we show that every financial structure satisfying WNMA is equivalent, in terms of financial possibilities and in terms of consumption equilibria, to another structure in which there are no useless portfolios (its reduced form). It is the purpose of future research to investigate the risk-sharing role of those redundant portfolios that cannot be dispensed with, without losing either of the properties equivalent or reduced. One can think of those portfolios as being useful.

The paper is organized as follows. In Section 2, we describe the financial exchange economy and introduce an equivalence relation on the set of financial structures. In Section 3 we state our results, define the reduced form of a financial structure, and provide proofs for the equilibrium existence theorems (Theorem 1 and 3). Section 3.3 provides sufficient conditions for Assumption F2 to hold. Section 4 is devoted to the proof of our main result (Theorem 2) as a consequence of a sharper result (Theorem 4). The appendix (Section 5) gathers the proofs of all the lemmas stated in the text.

2. The two-date model of a financial economy

2.1. Standard exchange economies

2 We consider the basic stochastic model with two dates: \( t = 0 \) (today) and \( t = 1 \) (tomorrow). At the second date, there is a nonempty finite set \( S := \{1, \ldots, S\} \) of states of nature, one of which prevails at time \( t = 1 \) and is only known at time \( t = 1 \). For convenience, \( s = 0 \) denotes the state of the world (known with certainty) at date 0 and we let \( S = \{0\} \cup S = \{0, 1, \ldots, S\} \). At each state, today and tomorrow, there is a spot market for a positive number \( \{\phi_s\}_{s \in S} \) of perfectly divisible perishable physical goods. A commodity is thus a couple \((h, s)\), specifying the physical good \( h \in 1, \ldots, \ell \) and the state \( s = 0, 1, \ldots, S \) at which it is available. Thus the commodity space is \( \mathbb{R}^L \), where \( L = \ell(1 + S) \) and we will use the notation \( x = (x(s))_{s \in S} \in \mathbb{R}^L \), where \( x(s) = (x_1(s), \ldots, x_\ell(s)) \in \mathbb{R}^\ell \), denotes the spot consumption at node \( s \in S \).
There is nonempty finite set \( I := \{1, \ldots, I\} \) of consumers, each of whom is endowed with a consumption set \( X_i \subset \mathbb{R}^L_i \), a preference correspondence \( P_i \), from \( \prod_{k \in I} X_k \) to \( X_i \), and an endowment vector \( e_i \in \mathbb{R}^L_i \). The set \( X_i \) is the set of her possible consumptions, and for \( x \in \prod_{k \in I} X_k \), \( P_i(x) \) is the set of consumption plans in \( X_i \) which are strictly preferred to \( x_i \) by consumer \( i \), given the consumption plans \( (x_{-i})_{j \neq i} \) of the other agents. The exchange economy can thus be summarized by \( \mathcal{E} = (I, S, (X_i, P_i, e_i)_{i \in I}) \).

We make the following standard assumptions C1-C6 on the consumption side. We denote by \( \mathcal{A}(\mathcal{E}) \) the set of attainable allocations of the economy, that is,

\[
\mathcal{A}_c = \{(x_i)_{i \in I} \in \prod_{i \in I} X_i : \sum_{i \in I} x_i = \sum e_i \}.
\]

**Consumption Assumption C** For every \( i \in I \) and for every \( x = (x_i)_{i \in I} \in \prod_i X_i \),

**C1 Consumption Sets:** \( X_i \) is a closed, convex, bounded below subset of \( \mathbb{R}^L_i \);

**C2 Continuity:** The correspondence \( P_i \), from \( \prod_{k \in I} X_k \) to \( X_i \), is lower semicontinuous \(^3\) with open values in \( X_i \) (for the relative topology of \( X_i \));

**C3 Convexity:** \( P_i(x) \) is convex;

**C4 Irreflexivity:** \( x_i \notin P_i(x) \);

**C5 Local Non-Satiation LNS:** \( \forall x \in \mathcal{A}_c \):

(a) \( \forall s \in S, \exists x'_i(s) \in \mathbb{R}^L_i \), \( (x'_i(s), x_i(-s)) \in P_i(x) \),\(^4\)

(b) \( \forall y_j \in P_i(x), (x_i, y_j) \subset P_i(x) \);

**C6 Consumption Survival CS:** \( e_i \in \text{int} X_i \).

We note that these assumptions are standard in a model with nonordered preferences; the assumptions on \( P_i \) are satisfied in particular when agents’ preferences are represented by utility functions that are continuous, strongly monotonic, and quasi-concave. An exchange economy \( \mathcal{E} \) satisfying Assumption C will be called **standard**.

2.2. Financial structures

Agents may operate financial transfers across states in \( \bar{S} \) (i.e. across the two dates and across the states of the second date) by exchanging finitely many assets \( j \in J := \{1, \ldots, J\} \). The assets are traded at the first date \( t = 0 \) and yield payoffs \( V_j^i(p) \) (for a given commodity price \( p \in \mathbb{R}^L \)) at the second date \( t = 1 \), contingent on the realization of the state of nature \( s \in S \). So, the payoff of asset \( j \) across tomorrow states is described by the mapping \( p \mapsto V_j^i(p) := (V_j^i(p))_{s \in S} \in \mathbb{R}^S \). The financial structure is described by the payoff matrix mapping \( V : p \mapsto V(p) \), where \( V(p) \) is the \( S \times J \)-matrix, whose columns are the payoffs \( V_j^i(p) \) \( (j = 1, \ldots, J) \) of the \( J \) assets. A portfolio \( z = (z_j) \in \mathbb{R}^J \) specifies quantities \( |z_j| \ (j \in J) \) of each asset \( j \), with the convention that the asset

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\(^3\) Let \( \Phi \) be a correspondence from \( X \) to \( Y \), that is, \( \Phi \) is a mapping from \( X \) to \( 2^Y \). Then \( \Phi \) is said to be lower semicontinuous (l.s.c.) at \( x_o \in X \), if for every open set \( V \subset Y \) such that \( \Phi(x_o) \cap V \neq \emptyset \), there exists an open neighborhood \( U \) of \( x_o \) in \( X \) such that \( \Phi(x) \cap V \neq \emptyset \) for all every \( x \in U \). The correspondence \( \Phi \) is said to be l.s.c. if it is l.s.c. at every point of \( X \). Finally, we denote by \( G(\Phi) := \{(x,y) \in X \times Y : y \in \Phi(x)\} \) the graph of \( \Phi \).

\(^4\)Given \( x_i \in X_i \) and \( s \in \bar{S} \), we denote \( x_i(-s) := (x_i(s'))_{s' \in S} \).
\( j \) is bought if \( z_j > 0 \) and sold if \( z_j < 0 \). Thus \( V(p)z \) is its random payoff across states at time \( t = 1 \), and \( V_i(p) \cdot z \) is its payoff if state \( s \) prevails. Each agent \( i \) is endowed with a portfolio set \( Z_i \subset \mathbb{R}^l \), which represents the constraints the agent faces. This general framework allows to address many economically relevant situations (see Elsinger and Summer (2001)). We summarize by \( \mathcal{F} = (\mathbf{I}, \mathbf{S}, \mathbf{J}, V, (Z_i)_{i \in \mathbf{I}}) \) the financial characteristics, referred to as the financial structure.

2.3. Equivalent financial structures

Given commodity and asset prices \( (p, q) \in \mathbb{R}^L \times \mathbb{R}^J \), the budget set of consumer \( i \) is

\[
B_i(p, q, \mathcal{E}, \mathcal{F}) = \{ (x_i, z_i) \in X_i \times Z_i : p(0) \cdot x_i(0) + q \cdot z_i \leq p(0) \cdot e_i(0) \\
p(s) \cdot x_i(s) \leq p(s) \cdot e_i(s) + V_i(p) \cdot z_i, \quad \forall s \in \mathbf{S} \}
\]

= \{ (x_i, z_i) \in X_i \times Z_i : p \Delta (x_i - e_i) \leq W(p, q)z_i \},

where \( W(p, q) \) denotes the total payoff matrix, that is, the \((1 + S) \times J\)-matrix \[
\begin{pmatrix}
-q \\
V(p)
\end{pmatrix}
\]. When there is no risk of confusion the budget set of consumer \( i \) will be denoted \( B_i(p, q) \). We now introduce the standard equilibrium notion in this model.

**Definition 1.** An equilibrium of the financial exchange economy \( (\mathcal{E}, \mathcal{F}) \) is a list \( (\bar{p}, \bar{x}, \bar{q}, \bar{z}) \in \mathbb{R}^L \times (\mathbb{R}^L)^J \times (\mathbb{R}^J)^J \) such that

(i) for every \( i, (\bar{x}_i, \bar{z}_i) \) maximizes the preference \( P_i \) in the budget set \( B_i(\bar{p}, \bar{q}) \), that is

\( (\bar{x}_i, \bar{z}_i) \in B_i(\bar{p}, \bar{q}) \) and \( B_i(\bar{p}, \bar{q}) \cap \{ P_i(\bar{x}) \times Z_i \} = \emptyset \),

(ii) [Market Clearing] \( \sum_{i \in \mathbf{I}} \bar{x}_i = \sum_{i \in \mathbf{I}} e_i \) and \( \sum_{i \in \mathbf{I}} \bar{z}_i = 0 \).

A consumption equilibrium of the financial exchange economy \( (\mathcal{E}, \mathcal{F}) \) is a list \( (\bar{p}, \bar{x}) \in \mathbb{R}^L \times (\mathbb{R}^L)^J \) such that there exist \( (\bar{q}, \bar{z}) \in \mathbb{R}^J \times (\mathbb{R}^J)^J \) and \( (\bar{p}, \bar{x}, \bar{q}, \bar{z}) \) is an equilibrium in \( (\mathcal{E}, \mathcal{F}) \).

We introduce an equivalence relation on the set of all financial structures defined on the same set of agents \( \mathbf{I} \) and the same set of states \( \mathbf{S} \). We will say that two financial structures are equivalent if they are indistinguishable in terms of consumption equilibria associated to the same standard exchange economy \( \mathcal{E} \). The intuition behind this definition is the following. Financial structures allow agents to transfer wealth across states of nature and thereby give them the possibility to enlarge their budget set. Hence if, regardless of the standard exchange economy \( \mathcal{E} \), equilibrium consumption allocations and equilibrium commodity price vectors are the same when agents carry out their financial activities through two different structures, then we say that these two financial structures are equivalent.

**Definition 2.** Consider two financial structures \( \mathcal{F} = (\mathbf{I}, \mathbf{S}, \mathbf{J}, V, (Z_i)_{i \in \mathbf{I}}) \) and \( \mathcal{F}' = (\mathbf{I}, \mathbf{S}, \mathbf{J}', V', (Z'_i)_{i \in \mathbf{I}}) \). We say that \( \mathcal{F} \) is equivalent to \( \mathcal{F}' \), denoted \( \mathcal{F} \sim \mathcal{F}' \), if for every standard exchange economy \( \mathcal{E} \), the financial exchange economies \( (\mathcal{E}, \mathcal{F}) \) and \( (\mathcal{E}, \mathcal{F}') \) have the same consumption equilibria.

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\(^5\)For every \( p = (p(s)), x = (x(s)) \in \mathbb{R}^L \), we denote by \( p \Delta x \) the vector \( (p(s) \cdot x(s))_{s \in \mathbf{S}} \).
3. Main results

3.1. Existence of equilibria and equivalent reduced forms of a financial structure

We make the following assumptions on the financial side of the economy. We denote by $AZ$ the asymptotic cone of a nonempty set $Z \subset \mathbb{R}^J$.

**F0.** The set $A_F(p) := A\left(\sum_{i \in I}(Z_i \cap \{V(p) \geq 0\})\right)$ does not depend on $p$ (hence denoted $A_F$ hereafter).

**F1.** For every $i \in I$, $Z_i$ is closed, convex, contains 0, and $V : \mathbb{R}^L \to \mathbb{R}^{S \times J}$ is continuous.

**F2:** One of the following two conditions holds:

(i) For all $i \in I$, $Z_i = P_i + K_i$ where $P_i$ is polyhedral convex, and $K_i$ is compact and convex.\(^7\)

(ii) Weak No Market Arbitrage (WNMA) [Hart (1974)]; for all $p \in \mathbb{R}^L$, if $\zeta_i \in AZ_i \cap \ker V(p)$ for all $i$ and $\sum_{i \in I} \zeta_i = 0$, then $\zeta_i \in AZ_i \cap -AZ_i \cap \ker V(p)$ for all $i$.

We say that the financial structure $F$ is **standard** if it satisfies the two assumptions $F0$ and $F1$. Assumption $F0$ allows to cover the cases of financial structures with nominal and numéraire assets (up to a modification of the payoff matrix in the case of numéraire assets); see next Section 3.2. However, $F0$ does not cover the general case of real assets as considered by Duffie and Shafer (1985) when the rank of $V(p)$ may change for which Hart (1974)'s counter-example applies and existence can be shown only generically. Assumption $F1$ is standard and needs no further comments. The convex polyhedral framework (condition (i) of Assumption $F2$) was considered in Aouani and Cornet (2009). Weak no market arbitrage was introduced by Hart (1974) and is standard in the literature on unbounded arbitrage. Propositions 2 and 3 provide sufficient conditions for assumption $F2$ to hold true. We now recall that equilibrium asset prices preclude unbounded arbitrage opportunities under the Local Non-Satiation Assumption $LNS$.

**Proposition 1.** Assume $LNS$ and that the portfolio sets $Z_i (i \in I)$ are closed and convex. If $(\bar{p}, \bar{x}, \bar{q}, \bar{z})$ is an equilibrium of the economy $(E, F)$, then $\bar{q}$ is arbitrage-free at $\bar{p}$, in the sense that there does not exist an agent $i$ and $\zeta_i \in AZ_i$ such that $W(\bar{p}, \bar{q})\zeta_i > 0$, that is

$$W(\bar{p}, \bar{q})(\bigcup_i AZ_i) \cap \mathbb{R}_+^S = \{0\}.$$  

We denote by $Q_F(p)$ the set of arbitrage-free asset prices at $p \in \mathbb{R}^L$.

We can now state our first result on the existence of an equilibrium of the financial economy $(E, F)$. Given the financial structure $F = (V, (Z_i)_{i \in I})$, we denote $Z_F := \langle \sum_{i \in I} Z_i \rangle$ the linear space spanned by $\sum_{i \in I} Z_i$, that is the space where financial activity takes place.

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\(^6\)The asymptotic cone of a nonempty subset $Z$ of $\mathbb{R}^J$ is the set $AZ := \{\lim_n \lambda^n z^n : (\lambda^n)_n \downarrow 0$ and $z^n \in Z$ for all $n\}$. As a consequence from the definition, one has $A(cZ) = AZ$ and we refer to Debreu (1959) for a general reference. When $Z$ is additionally assumed to be convex, then $AZ = 0^*(cZ)$, where $0^*(C) := \{\zeta \in \mathbb{R}^J : \zeta + C \subset C\}$ is the recession cone of the convex set $C \subset \mathbb{R}^J$ (see Rockafellar (1997)). When $Z$ is convex, the inclusion $0^*(Z) \subset AZ$ holds but may be strict when $Z$ is not closed.

\(^7\)We say that $Z \subset \mathbb{R}^J$ is a polyhedral convex set if it can be defined by finitely many linear inequalities, i.e., $Z := \{z \in \mathbb{R}^J : Bz \geq b\}$ for some $K \times J$-matrix $B$ and some $b \in \mathbb{R}^K$. 

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Theorem 1. The economy \((E, F)\) admits an equilibrium \((\bar{p}, \bar{x}, \bar{q}, \bar{z})\) such that \(\|\bar{p}(0)\| + \|\bar{q}\| = 1\) and \(\|\bar{p}(s)\| = 1\) for \(s \in S\) if it satisfies assumptions \(C, F_0, F_1,\) and \(F_2\) together with

Financial Survival FS: \(\forall i \in I, \forall p \in \mathbb{R}^L, p(0) = 0, \forall q \in cl[Q_F(p) \cap Z_F], q \neq 0, \exists \xi_i \in Z_i, q \cdot \xi_i < 0.\)

It is worthwhile to note that under the assumptions of Theorem 1 the set of admissible consumption allocations, \(\mathcal{A}_E\), is compact but the set of “admissible”\(^9\) portfolio allocations, \(\mathcal{A}_F(p, v)\), may not be bounded (it is clearly closed), where

\[
\mathcal{A}_F(p, v) := \{(z_i) : \forall i \in I, \forall v \in cl(V(p)z_i), \sum_{i \in I} z_i = 0\}.
\]

In order to circumvent this difficulty, the proof of Theorem 1 consists in two steps. The first step relies on the following result (Theorem 2) which associates an equivalent \(^{10}\) reduced form \(F'\) to the financial structure \(F\). The new financial structure \(F'\) will be reduced in the sense that \(\mathcal{A}_F(p, v)\) is compact. The second step of the proof consists in showing that \((E, F')\) admits an equilibrium (using the compactness property of \(F'\)).

Definition 3. The financial structure \(F = (I, S, J, V, (Z_i)_i)\) is said to be reduced if for every \(p \in \mathbb{R}^L\) and for every \(v = (v_i)_i \in (\mathbb{R}^S)^I\), the set \(\mathcal{A}_F(p, v)\) of admissible portfolio allocations is bounded.

Theorem 2. Let \(F = (I, S, J, V, (Z_i)_i)\) be a standard (i.e. satisfies \(F_0\) and \(F_1\)) financial structure satisfying \(F_2\). Then there exists a standard reduced financial structure \(F'\) such that:

(a) for every standard exchange economy \(E\), every consumption equilibrium of \((E, F')\) is a consumption equilibrium of \((E, F)\).

(b) \(F'\) satisfies the Survival Assumption FS if \(F\) satisfies the Survival Assumption FS.

The proof of Theorem 2 is given in Section 4 as a consequence of the sharper result Theorem 4.

Proof of Theorem 1. Let \((E, F)\) satisfy the assumptions of Theorem 1. The proof consists in two steps. First, since \(F\) is standard and satisfies \(F_2\), from Theorem 2 we obtain a standard reduced

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\(^9\)In Aouani and Cornet (2009), the financial survival assumption is made on prices \(q\) in \(cl[Q_F(p) \cap Z_F]\), so it seems to be stronger than the one used here. The two conditions are actually equivalent. Indeed \(cl[Q_F(p) \cap Z_F] = cl[Q_F(p) \cap Z_F]\), to see this, note that the inclusion \(cl[Q_F(p) \cap Z_F] \subset cl[Q_F(p) \cap Z_F]\) is immediate and it remains to show the converse inclusion. Let \(q \in cl[Q_F(p) \cap Z_F]\). Then \(q = \lim_n (q^n)\) for some sequence \((q^n)_n \subset Q_F(p)\). Since \(q \in Z_F\), one has \(q = \text{proj}_{Z_F} q = \lim_n \text{proj}_{Z_F} q^n\). The sequence \((\text{proj}_{Z_F} q^n)_n\) is clearly in \(Z_F\) and one easily checks that it is also in \(Q_F(p)\).

\(^{10}\)The set of admissible portfolio contains the constraints “\(V(p)z_i \geq v_i\) since they are satisfied for some \(v_i \in \mathbb{R}^S\) by all equilibrium portfolio allocations; indeed from agents’ budget constraints, at equilibrium one has \(\bar{p}(s) \cdot (\bar{x}_i(s) - e_i(s)) \leq V_i(\bar{p}) \cdot \bar{z}_i\) \((s \in S)\), and the left-hand side is bounded below when equilibrium commodity prices \(\bar{p}\) belongs to \(B_T(0, 1)\) and equilibrium consumption allocations are bounded (by assumption \(C\)); see Aouani and Cornet (2009).

\(^{11}\)For the existence problem (the proof of Theorem 1), we only need the property that every consumption equilibrium of \((E, F')\) is a consumption equilibrium of \((E, F)\) as stated in Theorem 2. We postpone the proof of the equivalence between \(F\) and its reduced form \(F'\) to Section 4.1, see Theorem 4, a sharper version of Theorem 2.
financial structure \( \mathcal{F}' \) satisfying \( \mathcal{FS} \) since \( \mathcal{F} \) satisfies \( \mathcal{FS} \). Second, by Theorem 2 (page 777) in Aouani and Cornet (2009), \((\mathcal{E}, \mathcal{F}')\) admits an equilibrium \((p', \bar{x}, q', z')\) such that \(\|p'(s)\| = 1\) for all \(s \in S\). The consumption equilibrium \((p', \bar{x})\) of \((\mathcal{E}, \mathcal{F}')\) is a consumption equilibrium of \((\mathcal{E}, \mathcal{F})\) by Theorem 2. Hence there exists \((q', \bar{z})\) such that \((p', \bar{x}, q', \bar{z})\) is an equilibrium of \((\mathcal{E}, \mathcal{F})\).

We end the proof by showing that \((\bar{p}, \bar{x}, \bar{q}, \bar{z})\) is an equilibrium of \((\mathcal{E}, \mathcal{F})\) for the prices \(\bar{p} = (\lambda p'(0), (p'(s))_{s \in S})\) and \(\bar{q} = \lambda q'\) with \(\lambda = 1/(\|p'(0)\| + \|q'\|)\) (so that \(\|\bar{p}(0)\| + \|\bar{q}\| = 1\)). Indeed, from Local Non-Satiation LNS (at \(s = 0\)), we deduce that \(\|p'(0)\| + \|q'\| > 0\), thus \(\lambda = 1/(\|p'(0)\| + \|q'\|)\) is well defined; moreover \((\lambda p'(0), (p'(s))_{s \in S}), \bar{x}, \lambda q', \bar{z})\) is an equilibrium of \((\mathcal{E}, \mathcal{F})\) (from the positive homogeneity property in \((p(0), q)\) of the budget constraint at \(t = 0\)).

**Remark 1.** It is worth mentioning that neither Theorem 1 nor Theorem 2 assumes that \(\text{rank} \ V(p) = J\) for every \(p\). This assumption is standard in the unconstrained case \((Z_i = \mathbb{R}^J \quad \text{for all} \quad i)\) since there is no loss of generality; indeed one can “reduce” the financial structure \(V\) by eliminating the redundant assets and associate a new financial structure \(V'\) which is “equivalent” to the previous one in the sense of Definition 2.

However this process (of eliminating redundant assets) is no longer possible in the constrained case but Theorem 2 provides an alternative way similar the procedure of Balasko et al. (1990) when the \(Z_i'\)s are vector spaces.

### 3.2. Existence of equilibria in the nominal and numéraire case

If the financial structure \(\mathcal{F}\) is nominal, the matrix \(V(p)\) of financial payoffs does not depend on the commodities price vector \(p\) and is denoted \(R\).

A numéraire asset is defined as follows. Let us choose a commodity bundle \(\nu \in \mathbb{R}^\ell\), a typical example being \(\nu = (0, \ldots, 0, 1)\), when the \(\ell\)-th good is chosen as numéraire. A numéraire asset \(j\) is a real asset which delivers \(R^j_1(\in \mathbb{R})\) units of the bundle \(\nu\), i.e., the commodity bundle \(A^j_\nu = R^j_1 \nu \in \mathbb{R}^\ell\) at state \(s\) of date \(t = 1\) if this state \(s\) prevails. Thus the payoff at state \(s\) is \((V_\nu)_j(p) = (p(s) \cdot \nu)R^j_1\) for the commodity price \(p = (p(s)) \in \mathbb{R}^L\). For a numéraire financial structure, i.e., all the assets are numéraire assets (for the same commodity bundle \(\nu\)), we denote \(R\) the \(S \times J\)-matrix with entries \(R^j_1\) and, for \(p \in \mathbb{R}^L\), we denote \(V_\nu(p)\) the associated \(S \times J\)-payoff-matrix, which has for entries \((V_\nu)_j(p) = (p(s) \cdot \nu)R^j_1\).

In the nominal case, the set \(Q(p)\) of arbitrage-free prices, that is, the set of asset prices \(q\) satisfying

\[
\begin{bmatrix} -q \\ R \end{bmatrix} \bigcup_i A Z_i \cap \mathbb{R}^S_+ = \{0\} \tag{3.1}
\]

does not depend on the price \(p\), hence is simply denoted \(Q_R\). In the numéraire case, under the Desirability Assumption (made in \(\text{FN(ii)}\) below, if \((\bar{p}, \bar{x}, \bar{q}, \bar{z})\) is an equilibrium, then \(\bar{p}(s) \cdot \nu > 0\) for all \(s \in S\) (see the proof of Lemma 2 in Aouani and Cornet (2009)), hence \(Q(\bar{p}) = Q_R\) as defined above by (3.1). Thus, every equilibrium asset price \(\bar{q}\) belongs to \(Q_R\) (by Proposition 1) in the nominal case and in the numéraire case.

To state our second result, we need the following general assumptions on the financial side. We refer to Aouani and Cornet (2009) for a thorough discussion of these assumptions.
FN0: The financial structure $\mathcal{F}$ is either (i) nominal, i.e., $V(p) = R$ is independent of $p$, or

(ii) numéraire, i.e., $V(p) = V_s(p)$ for some $\nu \in \mathbb{R}^J$, for every agent $i$ the correspondence $P_i$ has an open graph, and the commodity bundle $\nu \in \mathbb{R}^J$ is desirable at every state $s \in S$, i.e., for all $x \in \mathcal{A}(E)$, for all $t > 0$, $(x(s) + tv, x_i(-s)) \in P_i(x)$;

Theorem 3. The economy $(E, \mathcal{F})$ admits an equilibrium $(\bar{p}, \bar{x}, \bar{q}, \bar{\xi})$ such that $||\bar{p}(0)|| + ||\bar{q}|| = 1$ and $||\bar{p}(s)|| = 1$ for $s \in S$ if it satisfies assumptions C, FN0, F1, F2, together with

FNS: $\forall i \in I, \forall q \in \text{cl} [Q_R \cap Z_F]$, $q \neq 0$, $\exists \xi_i \in Z_i$, $q \cdot \xi_i < 0$.

Proof. In the nominal case the proof of Theorem 3 is a straightforward consequence of Theorem 1. Consider now the numéraire case and let $(\mathcal{E}, \mathcal{F})$ be the financial economy with numéraire assets satisfying FN0 (Part (ii) with numéraire assets), F1, and F2. The proof consists in two steps. First we consider a modified financial structure $\mathcal{F}^e$ of $\mathcal{F}$ so that (i) $\mathcal{F}^e$ satisfies F0, F1, F2, and FNS, and (ii) the equilibria of $(\mathcal{E}, \mathcal{F}^e)$ are also equilibria of the original financial economy $(\mathcal{E}, \mathcal{F})$.

First, we define $\mathcal{F}^e = (V^e, (Z_i))$ for $\varepsilon > 0$, by taking the same portfolio sets $Z_i$ as for $\mathcal{F}$ and

$$V^e(p) = \begin{bmatrix} \max\{\varepsilon, p(1) - \nu\} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \max\{\varepsilon, p(S) - \nu\} \end{bmatrix} + R.$$

The financial structure $\mathcal{F}^e$ satisfies F0, F1, F2, and FNS since the financial structure $\mathcal{F}$ satisfies assumptions FN0, F1, F2, and FNS. Indeed, $\{V^e(p) \geq 0\} = \{R \geq 0\}$ for every $p \in \mathbb{R}^J$, hence $\mathcal{F}^e$ satisfies F0. Assumptions F1 and F2 are obviously satisfied, and $\mathcal{F}^e$ satisfies FNS since $\mathcal{F}$ satisfies FNS and $Q_{\mathcal{F}^e}(p) = Q_R$ for every $p$.

Second, for $\varepsilon > 0$ small enough, every equilibrium $(\bar{p}, \bar{x}, \bar{q}, \bar{\xi})$ of $(\mathcal{E}, \mathcal{F}^e)$ such that $||\bar{p}(s)|| = 1$ for $s \in S$ is an equilibrium of the economy $(\mathcal{E}, \mathcal{F})$ as shown by Aouani and Cornet (2009). This ends the proof of Theorem 3.

We can now state some consequences to Theorem 3. The following corollary extends to the case of consumers with non-ordered preferences the existence results of Cass (1984), Duffie (1987), Werner (1985), and Siconolfi (1989) in the nominal case, Geanakoplos and Polemarchakis (1986) in the numéraire case, and Radner (1972) in the general case of real assets.

Corollary 1. The economy $(\mathcal{E}, \mathcal{F})$ admits an equilibrium under Assumption C, F1 in each of the following cases:

- FN0, F2, and $0 \in \text{int} Z_i$ for all $i$.
- (Cass (1984), Duffie (1987), Werner (1985)) $\mathcal{F}$ consists of nominal assets and $Z_i = \mathbb{R}^J, (i \in I)$.
- (Geanakoplos and Polemarchakis, 1986) $\mathcal{F}$ consists of numéraire assets, satisfies FN0 (ii), and $Z_i = \mathbb{R}^J, (i \in I)$.
- (Radner, 1972) $\mathcal{F}$ satisfies F0 and for every $i \in I$, $Z_i = \{z \in \mathbb{R}^J : ||z|| \leq r_i\}, \text{for some } r_i > 0.$
- (Radner, 1972) $\mathcal{F}$ satisfies F0 and for every $i \in I$, $Z_i = z_i + \mathbb{R}^J_+, \text{for some } z_i \in -\mathbb{R}^J_+.$
- (Siconolfi, 1989) $\mathcal{F}$ consists of nominal assets, FNS holds and $AZ_i \cap \ker R = \{0\}$ for all $i \in I$.  


3.3. Examples of restrictions satisfying Assumption F2

As shown by the following Propositions 2 and 3, Assumption F2 holds true in many situations. Indeed, F2 is fulfilled when restrictions on portfolio choices are given by a finite number of linear inequalities, that is, when all portfolio sets are finite intersections of half spaces. In particular, F2 is fulfilled when portfolio sets are linear subspaces, when portfolio sets are unconstrained, or when there is an exogenous bound on portfolio short sales. Furthermore, Assumption F2 holds true under WNMA (Hart, 1974) that is when, for all \( p \), the family \( \{AZ_i \cap \ker V(p), i \in I\} \) is weakly positively semi-independent.\(^{11}\) In particular, Assumption F2 holds true under the No Unbounded Arbitrage condition (NUBA) (Page, 1987) that is when, for all \( p \), the family \( \{AZ_i \cap \ker V(p), i \in I\} \) is positively semi-independent.\(^{12}\) under Siconolfi (1989)’s assumption \( \{AZ_i \cap \ker V(p) = \{0\} \) for all \( i \in I \), when portfolio sets are bounded, or when there are no redundant assets i.e. \( \text{rank} V = J \).

Proposition 2. Assumption F2(i) holds true under anyone of the following conditions.

1. For all \( i \in I \), \( Z_i = \mathbb{R}^I \) (unconstrained portfolios).
2. For all \( i \in I \), \( Z_i \) is a linear subspace (linear equality constraints).
3. For all \( i \in I \), \( Z_i = \mathbb{Z}_+ + \mathbb{R}^I \), for some \( \mathbb{Z}_+ \subset -\mathbb{R}^I \) (exogenous bounds on short sales).
4. For all \( i \in I \), \( Z_i \) is polyhedral (linear inequality constraints).
5. For all \( i \in I \), \( Z_i = B_j(0, 1) \) (bounded portfolio sets).

The proof of Proposition 2 is immediate and therefore is omitted.

Proposition 3. Assumption F2(ii) holds true under each of the following conditions.

1. There are no redundant assets i.e. for all \( p \in \mathbb{R}^L \), \( \text{rank}(V(p)) = J \), or equivalently, \( \ker V(p) = \{0\} \).
2. For all \( p \in \mathbb{R}^L \) and for all \( i \in I \), \( AZ_i \cap \ker V(p) = \{0\} \).
3. For all \( p \in \mathbb{R}^L \), \( A(\sum_{i \in I} Z_i \cap \{V(p) \geq 0\}) \cap -A(\sum_{i \in I} Z_i \cap \{V(p) \geq 0\}) = \{0\} \).
4. For all \( p \in \mathbb{R}^L \), \( A(\sum_{i \in I} Z_i \cap \ker V(p)) \cap -A(\sum_{i \in I} Z_i \cap \ker V(p)) = \{0\} \).
5. For all \( p \in \mathbb{R}^L \), \( \left(\sum_{i \in I} AZ_i \cap \{V(p) \geq 0\}\right) \cap -\left(\sum_{i \in I} AZ_i \cap \{V(p) \geq 0\}\right) = \{0\} \).
6. For all \( p \in \mathbb{R}^L \), \( \left(\sum_{i \in I} AZ_i \cap \ker V(p)\right) \cap -\left(\sum_{i \in I} AZ_i \cap \ker V(p)\right) = \{0\} \).
7. For all \( p \in \mathbb{R}^L \), the family \( \{AZ_i \cap \{V(p) \geq 0\} : i \in I\} \) is positively semi-independent.
8. For all \( p \in \mathbb{R}^L \), the family \( \{AZ_i \cap \ker V(p) : i \in I\} \) is positively semi-independent.
9. For all \( p \in \mathbb{R}^L \), the family \( \{AZ_i \cap \{V(p) \geq 0\} : i \in I\} \) is weakly positively semi-independent.

The proof of Proposition 3 is immediate and therefore is omitted.

\(^{11}\)A finite collection \( \{C_i, i \in I\} \) of nonempty convex cones of \( \mathbb{R}^n \) is weakly positively semi-independent if \( c_i \in C_i \) for all \( i \in I \) and \( \sum_{i \in I} c_i = 0 \) imply that for all \( i \in I \), \( c_i \in C_i \cap -C_i \).

\(^{12}\)A finite collection \( \{C_i, i \in I\} \) of nonempty convex cones in \( \mathbb{R}^n \) is positively semi-independent if \( c_i \in C_i \), for all \( i \in I \) and \( \sum_{i \in I} c_i = 0 \), imply that for all \( i \in I \), \( c_i = 0 \).
4. Proof of Theorem 2

4.1. A sharper result

We introduce a weaker assumption than \( F_2 \).

**Closedness:** \( \mathcal{F} \) is closed, that is, for all \( p \in \mathbb{R}^L \), the set \( G_p(p) \) is closed, where

\[
G_p(p) := \{ (V(p)z_1, \ldots, V(p)z_I, \sum_{i \in I} z_i) \in (\mathbb{R}^J)^I \times \mathbb{R}^J : (z_i)_{i \in I} \in \prod_{i} Z_i \}.
\]

It is important to notice that every reduced financial structure is closed. Indeed, Proposition 4 asserts that Closedness is weaker than \( F_2 \) – therefore all the conditions listed in Proposition 2 and 3 are sufficient conditions for the Closedness Assumption to hold – and that every reduced financial structure satisfies WNMA.

Let \( \mathcal{F} = (V, (Z_i)_i) \) be a standard and closed financial structure. We consider the financial structure \( \mathcal{F}_\pi \) which has the same payoff matrix as \( \mathcal{F} \) and the portfolio sets \( cl\pi Z_i (i \in I) \) where \( \pi \) is the orthogonal projection mapping of \( \mathbb{R}^l \) on the orthogonal space to \( L_\mathcal{F} := A_\mathcal{F} \cap -A_\mathcal{F} \). We recall that \( Z_\mathcal{F} := \cup_i Z_i >, Z_\mathcal{F}_\pi := \cup_i cl\pi Z_i > \) and the definition of \( \mathcal{F}_\pi \) can be summarized by \( \mathcal{F}_\pi = (V, (cl\pi Z_i)_i) \), where

\[
\pi = \text{proj}_{(L_\mathcal{F})^\perp}, L_\mathcal{F} := A_\mathcal{F} \cap -A_\mathcal{F}, \text{ and } A_\mathcal{F} := A\left(\sum_{i \in I} (Z_i \cap \{V(p) \geq 0\})\right) \subset Z_\mathcal{F}.
\]

Note that \( L_\mathcal{F} \subset \ker V(p) \) for all \( p \in \mathbb{R}^L \). We will use extensively the following properties\(^{13} \) of the linear mapping \( \pi \): for all \( (p, q, z) \in \mathbb{R}^L \times \mathbb{R}^I \times \mathbb{R}^l \),

\[
q \cdot \pi z = \pi q \cdot z = \pi q \cdot z, \quad \ker \pi = L_\mathcal{F}, \quad V(p)\pi z = V(p)z, \quad \text{hence } W(p, q)\pi z = W(p, q)z. \quad (4.1)
\]

Given the financial structure \( \mathcal{F} \) and given \( p \in \mathbb{R}^L \), we denote

\[
\mathcal{V}_p(\mathcal{F}) := \{(V(p)z_1, \ldots, V(p)z_I, (z_i)_{i \in I} \in \prod_{i} Z_i, \sum_{i} z_i = 0 \}.
\]

Theorem 2 is a direct consequence of the following theorem.

**Theorem 4.** Let \( \mathcal{F} = (V, (Z_i)_i) \) be a standard financial structure.

(a) \( \mathcal{F}_\pi \) satisfies property \( Q \) below.

\[
Q: \forall p \in \mathbb{R}^L, \text{cl}\left[Q_{\mathcal{F}_\pi}(p) \cap Z_{\mathcal{F}_\pi}\right] \subset \text{cl}\left[Q_{\mathcal{F}}(p) \cap Z_{\mathcal{F}}\right].
\]

Hence, if \( \mathcal{F} \) satisfies \( FS \) then so does \( \mathcal{F}_\pi \).

(b) If \( \mathcal{F} \) is closed then \( \mathcal{F}_\pi \) is standard, reduced, and \( \mathcal{V}_p(\mathcal{F}) = \mathcal{V}_p(\mathcal{F}_\pi), \forall p \in \mathbb{R}^L \).

---

\(^{13}\)The first equality comes from the fact that \( \pi \cdot \pi z = \pi q \cdot z \), since \( \pi q \in \text{Im} \pi \) and \( z - \pi z \in \ker \pi = (\text{Im} \pi)^\perp \) since \( \pi \) is an orthogonal projection mapping; then by symmetry \( q \cdot \pi z = \pi q \cdot \pi z = \pi q \cdot z \). The second one holds since \( z - \pi z \in \ker \pi = L_\mathcal{F} \) and \( L_\mathcal{F} := A_\mathcal{F} \cap -A_\mathcal{F} \subset \{V(p) \geq 0\} \cap -\{V(p) \geq 0\} = \ker V(p) \).
(c) If $\mathcal{F} = \left( V, (Z_i)_{i} \right)$ satisfies $\mathcal{V}_{p}(\mathcal{F}) = \mathcal{V}_{p}(\mathcal{F}_{\pi}), \forall p \in \mathbb{R}^{L}$ (hence in particular if $\mathcal{F}$ is closed), then for every standard exchange economy $\mathcal{E}$, for every equilibrium $(\tilde{p}, \tilde{q}, \tilde{x}, \tilde{y})$ of $(\mathcal{E}, \mathcal{F}_{\pi})$, there exists $z^* \in \prod_{i} Z_{i}$ such that $(\tilde{p}, \tilde{q}, \tilde{x}, z^*)$ is an equilibrium of $(\mathcal{E}, \mathcal{F})$.

(d) If $\mathcal{F} = \left( V, (Z_i)_{i} \right)$ satisfies $\mathcal{V}_{p}(\mathcal{F}) = \mathcal{V}_{p}(\mathcal{F}_{\pi}), \forall p \in \mathbb{R}^{L}$ (hence in particular if $\mathcal{F}$ is closed), the financial structures $\mathcal{F}$ and $\mathcal{F}_{\pi}$ are equivalent if $\mathcal{F}$ additionally has a riskless asset, i.e.,

\[ \text{FA: for all } p \in \mathbb{R}^{L} \text{ and for all } i \text{, there exists } \zeta_{i,p} \in AZ_{i} \text{ such that } V(p)\zeta_{i,p} \geq 0. \]

The proof of Theorem 4 is given in Section 4.2.

4.2. Proof of Theorem 4

4.2.1. Proof of Part (a) of Theorem 4

$\mathcal{F}_{\pi}$ satisfies property Q: We claim that, for every $p \in \mathbb{R}^{L}$,

\[ Q_{\mathcal{F}_{\pi}}(p) \cap Z(\mathcal{F}_{\pi}) \subset Q_{\mathcal{F}}(p) \cap Z_{\mathcal{F}} \tag{4.2} \]

First, we show that $Q_{\mathcal{F}_{\pi}}(p) \cap Z(\mathcal{F}_{\pi}) \subset Q_{\mathcal{F}_{\pi}}(p) \cap \text{Im} \pi \subset Q_{\mathcal{F}}(p)$. The first inclusion is a consequence of the fact that $Z(\mathcal{F}_{\pi}) \subset \text{Im} \pi$. We prove the second inclusion by contradiction. Assume that there is some $q \in Q_{\mathcal{F}_{\pi}}(p) \cap \text{Im} \pi$ such that $q \notin Q_{\mathcal{F}}(p)$. Then there exists $i \in I$ and $\zeta_{i} \in AZ_{i}$ such that $W(p,q)\zeta_{i} > 0$. But $\pi_{\zeta_{i}} \in \pi(AZ_{i}) \subset A(\pi Z_{i})$ (from Rockafellar (1997)) and (since $q \in \text{Im} \pi$ implies $q = \pi q$), from (4.1) $W(p,q)(\pi_{\zeta_{i}}) = W(p,q)\zeta_{i} > 0$, which contradicts the fact that $q \in Q_{\mathcal{F}_{\pi}}(p)$. This ends the proof of the two inclusions. We end the proof of (4.2) by showing that $Z(\mathcal{F}_{\pi}) \subset Z_{\mathcal{F}}$. Indeed, let $y \in Z(\mathcal{F}_{\pi})$, then $y = \pi_{\zeta}$ for some $\zeta \in Z_{\mathcal{F}}$ and $y = \pi_{\zeta} = \pi_{\zeta} - \zeta + \zeta \in \ker \pi + Z_{\mathcal{F}} \subset Z_{\mathcal{F}}$ since $\ker \pi \subset L_{\mathcal{F}} \subset Z_{\mathcal{F}}$. The inclusion (4.2) implies that $\mathcal{F}_{\pi}$ satisfies property Q.

If $\mathcal{F}$ satisfies FS then so does $\mathcal{F}_{\pi}$: Let $i \in I, p \in \mathbb{R}^{L}$ such that $p(0) = 0$, and $q \in \text{cl}[Q_{\mathcal{F}_{\pi}}(p) \cap Z_{\mathcal{F}_{\pi}}]$. By property Q, we have $q \in \text{cl}[Q_{\mathcal{F}}(p) \cap Z_{\mathcal{F}}]$. Since $\mathcal{F}$ satisfies FS, there exists $z_{i} \in Z_{i}$ such that $q - z_{i} < 0$. Therefore the portfolio $\pi z_{i} \in \pi Z_{i} \subset \text{cl} \pi Z_{i}$ and $q - \pi z_{i} = q - z_{i}$ (because $q \in Z_{\mathcal{F}_{\pi}} \subset \text{Im} \pi$ and $\pi$ is an orthogonal projection). Hence $q - \pi z_{i} < 0$.

4.2.2. Proof of Part (b) of Theorem 4

First we state two lemmas that shall be useful in the sequel.

**Lemma 1.** Let $\mathcal{F} = \left( V, (Z_i)_{i} \right)$ be standard and closed. For all $p \in \mathbb{R}^{L}$ and for all $(\tilde{y}_{i})_{i} \in \prod_{i} \text{cl} \pi Z_{i}$ such that $\sum_{i} \tilde{y}_{i} = 0$, one has

\[ \sum_{i} (\text{cl} \pi Z_{i} \cap \{V(p) \geq V(p)\tilde{y}_{i}\}) \subset \sum_{i} (Z_{i} \cap \{V(p) \geq V(p)\tilde{y}_{i}\}). \]

Hence $L_{\mathcal{F}_{\pi}} \subset L_{\mathcal{F}}$.

**Lemma 2.** If $L_{\mathcal{F}} := A_{\mathcal{F}} \cap -A_{\mathcal{F}} = \{0\}$, then $\mathcal{F}$ is reduced.

$\mathcal{F}_{\pi}$ is standard: $\mathcal{F}_{\pi}$ obviously satisfies F1 and it remains to show that $A_{\mathcal{F}_{\pi}}(p)$ does not depend on $p$. We show that for all $p \in \mathbb{R}^{L}$, $\pi(A_{\mathcal{F}}(p)) = A_{\mathcal{F}_{\pi}}(p)$ and the desired result follows from the fact that $A_{\mathcal{F}}(p)$ is independent of $p$. For the first inclusion: we have

\[ \pi(A_{\mathcal{F}}(p)) \subset A_{\pi} \sum_{i} (Z_{i} \cap \{V(p) \geq 0\}) = A \sum_{i} \pi Z_{i} \cap \{V(p) \geq 0\} \subset A_{\mathcal{F}_{\pi}}(p). \]
The first inclusion follows from $\pi(AC) \subset A(\pi C)$ (see Rockafellar (1997)). The equality comes from (5.1). The last inclusion comes from “$C_1 \subset C_2 \Rightarrow AC_1 \subset AC_2$.”

For the converse inclusion: From Lemma 1, taking $\hat{y}_i = 0$ for all $i \in I$, and then the asymptotic cones of both sides of the inclusion we get $A_{\mathcal{F}_A}(p) \subset A(p)$. Thus $\pi(A_{\mathcal{F}_A}(p)) \subset \pi(A(p))$. It suffices to notice that $A_{\mathcal{F}_A}(p) \subset \text{Im} \pi$ to conclude that $A_{\mathcal{F}_A}(p) \subset \pi(A(p))$.

$\mathcal{F}_\pi$ is reduced: First, we claim that $L_{\mathcal{F}_A} \subset \mathcal{L}_{\mathcal{F}} \cap \text{Im} \pi$. Indeed, we clearly have $L_{\mathcal{F}_A} \subset \text{Im} \pi$ since $\sum_{i \in I}(\text{cl} \pi Z_i \cap \{V(p) \geq 0\}) \subset \text{Im} \pi$, and by Lemma 1, $L_{\mathcal{F}_A} \subset \mathcal{L}_{\mathcal{F}}$. This ends the proof of the claim. Since $\mathcal{L}_{\mathcal{F}} = \ker \pi$, then from the above claim, we get $L_{\mathcal{F}_A} \subset \mathcal{L}_{\mathcal{F}} \cap \text{Im} \pi = \ker \pi \cap \text{Im} \pi = \{0\}$. Using Lemma 2 we conclude that $\mathcal{F}_\pi$ is reduced. This ends the proof of Part (a) of Theorem 4.

Let $p \in \mathbb{R}^L$.

$V_p(\mathcal{F}) \subset V_p(\mathcal{F}_\pi)$: If $z_i \in Z_i (i \in I)$ are such that $\sum_{i \in I}z_i = 0$, then $y_i = \pi z_i \in \text{cl} \pi Z_i (i \in I)$ satisfy $\sum_{i \in I}y_i = \sum_{i \in I} \pi z_i = \pi(\sum_{i \in I}z_i) = 0$, and for each $i \in I$, $V(p)y_i = V(p)(\pi z_i - z_i) + V(p)z_i = V(p)z_i$ since $\ker \pi \subset \ker(p)$ (recall that $\ker \pi = L_{\mathcal{F}_A} \subset \ker V(p)$).

$V_p(\mathcal{F}_\pi) \subset V_p(\mathcal{F})$: Let $y := (y_i) \in \prod_i \text{cl} \pi Z_i$ be such that $\sum_{i \in I}y_i = 0$. Then, by Lemma 1,

$$0 = \sum_{i \in I}y_i \in \sum_{i \in I}\left(\text{cl} \pi Z_i \cap \{V(p) \geq V(p)y_i\}\right) \subset \sum_{i \in I}\left(Z_i \cap \{V(p) \geq V(p)y_i\}\right).$$

Hence $0 = \sum_{i \in I}z_i$ for some $z_i \in Z_i \cap \{V(p) \geq V(p)y_i\}$, that is, $z_i \in Z_i$ and $V(p)(z_i - y_i) \geq 0$ for every $i$. Noticing that $\sum_{i \in I}(z_i - y_i) = 0$, we get $\sum_{i \in I}(V(p)(z_i - y_i) = 0$ and we conclude that $V(p)z_i - V(p)y_i = 0$ for every $i$.

4.2.3. Proof of Part (c) of Theorem 4

First we need a claim.

Claim 4.1. If $(q, y)$ is arbitrage-free at $p$ in $\mathcal{F}_\pi$ and $\sum_{i \in I}y_i = 0$, then there exists a mutually compatible portfolio allocation $z^* \in \prod_i Z_i$ such that $W(p, q)y_i = W(p, \pi q)z^*_i$ for every $i \in I$ and $(\pi q, z^*)$ is arbitrage-free at $p$.

Proof. Let $(q, (y_i)) \in \mathbb{R}^L \times (\prod_i \text{cl} \pi Z_i)$ be arbitrage-free in $\mathcal{F}_\pi$ such that $\sum_{i \in I}y_i = 0$. Then, $(V(p)y_1, \ldots, V(p)y_I) \in V_{\mathcal{F}_A}(p)$. Hence, by assumption $(V_{\mathcal{F}_A}(p) = V_{\mathcal{F}}(p))$, for each $i \in I$, there exists $z^*_i \in Z_i$ such that $\sum_{i \in I}z^*_i = 0$ and $V(p)z^*_i = V(p)y_i$.

Now, we show that, for every $i$, $\pi q \cdot z^*_i = \pi q \cdot y_i$ (which, by (4.1), is equal to $q \cdot y_i$). Let us first note that it suffices to show that for every $i \in I$, $-\pi q \cdot z^*_i \leq -\pi q \cdot y_i$. In this case $-\pi q \cdot \sum_{i \in I}z^*_i - \sum_{i \in I}y_i = 0$ implies $\pi q \cdot z^*_i = \pi q \cdot y_i$, for every $i$. Suppose, that for some $i$, $-\pi q \cdot z^*_i > -\pi q \cdot y_i$. Since $V(p)z^*_i = V(p)y_i$, one has $W(p, \pi q)z^*_i > W(p, q)y_i$. Then from (4.1) $W(p, q)z^*_i > W(p, q)\pi y_i$. Moreover $\pi z^*_i \in \pi Z_i \subset \text{cl} \pi Z_i$. It thus suffices to show that $\pi y_i = y_i$ to get $W(p, q)\pi z^*_i > W(p, q)y_i$ which would contradict the assumption that $(q, y)$ is arbitrage-free in $\mathcal{F}_\pi$. Since $y_i \in \text{cl} \pi Z_i$, one has $y_i = \lim_n \pi y^*_i$ with $y^*_i \in Z_i$. Then $\pi y_i = \pi \lim_n \pi y^*_i = \lim_n \pi (\pi y^*_i) = \lim_n \pi y^*_i = y_i$.

\footnote{$(\bar{q}, \bar{z})$ is arbitrage-free at $\bar{p}$ if for every $i \in I$, there is no $z_i \in Z_i$ such that $W(\bar{p}, \bar{q})z_i > W(\bar{p}, \bar{q})\bar{z}_i$.}
Finally, we show that \((pq, z^*)\) is arbitrage-free at \(p\) in \(\mathcal{F}\). Assume that for some \(i \in \mathbf{I}\), there exists \(z^*_i \in Z_i\) such that \(W(p, pq\bar{z}^*_i) > W(p, pqz^*_i)\). Then from (4.1) \(W(p, q)\pi z^*_i > W(p, q)y_i\). A contradiction to the fact that \((q, y)\) is arbitrage-free at \(p\) in \(\mathcal{F}_\pi\). This ends the proof of the claim. ■

We show that if \((\mathcal{E}, \mathcal{F}_\pi)\) has an equilibrium \((\bar{p}, \bar{x}, \bar{q}, \bar{y})\), then there exists \(z^* \in \prod_i Z_i\) such that \((\bar{p}, \bar{x}, \pi \bar{q}, z^*)\) is an equilibrium of \((\mathcal{E}, \mathcal{F})\). Let \((\bar{p}, \bar{x}, \bar{q}, \bar{y})\) be an equilibrium in \((\mathcal{E}, \mathcal{F}_\pi)\). Then \((\bar{q}, \bar{y})\) is arbitrage-free at \(\bar{p}\) in \(\mathcal{F}\) (see Angeloni and Cornet (2006)), and by the above Claim 4.1, for each \(i\), there exists \(z^*_i \in Z_i\) such that \(W(\bar{p}, \pi \bar{q})z^*_i = W(\bar{p}, \bar{q})y_i\), \(\sum_i z^*_i = 0\), and \((\pi \bar{q}, z^*)\) is arbitrage-free at \(\bar{p}\) in \(\mathcal{F}\). We show that \((\bar{p}, \pi \bar{q}, \bar{x}, z^*)\) is an equilibrium of \((\mathcal{E}, \mathcal{F})\). First, from \(W(\bar{p}, \pi \bar{q})z^*_i = W(\bar{p}, \bar{q})y_i\) for each \(i \in \mathbf{I}\), we conclude that \((\bar{x}_i, z^*_i) \in B_i(\bar{p}, \pi \bar{q}, \mathcal{F})\) since \((\bar{x}_i, y_i) \in B_i(\bar{q}, \bar{y}, \mathcal{F}_\pi)\). To complete the proof, we need only show that for each \(i \in \mathbf{I}\),

\[
B_i(\bar{p}, \pi \bar{q}, \mathcal{F}) \cap (P_i(\bar{x}) \times Z_i) = \emptyset.
\]

Since \((\bar{p}, \bar{x}, \bar{q}, \bar{z})\) is an equilibrium of \((\mathcal{E}, \mathcal{F}_\pi)\), we have

\[
B_i(\bar{p}, \bar{q}, \mathcal{F}_\pi) \cap (P_i(\bar{x}) \times \text{cl} \pi Z_i) = \emptyset.
\]

In view of the above, the proof will be completed if we show that if \((x_i, z_i) \in B_i(\bar{p}, \pi \bar{q}, \mathcal{F})\), then \((x_i, \pi z_i) \in B_i(\bar{p}, \bar{q}, \mathcal{F}_\pi)\). This is true since \(W(\bar{p}, \pi \bar{q})z_i = W(\bar{p}, \bar{q})\pi z_i\) by (4.1). ■

4.2.4. Proof of Part (d) of Theorem 4

Let \(\mathcal{F} = (V, (Z_i))\) be a standard financial structure satisfying \(\text{FA}\) and such that \(\mathcal{V}_p(\mathcal{F}) = \mathcal{V}_p(\mathcal{F}_\pi), \forall p \in \mathbb{R}^L\). We show that \(\mathcal{F}\) and \(\mathcal{F}_\pi\) are equivalent. By Part (c) of Theorem 4, for every standard exchange economy \(\mathcal{E}\), every consumption equilibrium of \((\mathcal{E}, \mathcal{F}_\pi)\) is a consumption equilibrium of \((\mathcal{E}, \mathcal{F})\). To end the proof of Part (d) of Theorem 4, we show that for every standard exchange economy \(\mathcal{E}\), if \((\mathcal{E}, \mathcal{F})\) has an equilibrium \((p^*, x^*, q^*, y^*)\), then \((p^*, x^*, \pi q^*, \pi z^*)\) is an equilibrium of \((\mathcal{E}, \mathcal{F}_\pi)\). We need two Lemmas.

**Lemma 3.** Let \(\mathcal{F} = (V, (Z_i))\) be a standard financial structure satisfying \(\text{FA}\). If \((\bar{q}, \bar{z})\) is arbitrage-free at \(\bar{p}\) in \(\mathcal{F}\) then \(\bar{q} \in \left(\text{A}_1 \sum_i (Z_i \cap \{V(\bar{p}) \geq 0\})\right)^0\). Hence \(\bar{q} \in \mathcal{L}^+_T\).

**Lemma 4.** Assume that for all \(s \in \bar{S}\), \(p(s) \neq 0\) and for all \(i \in \mathbf{I}\), \(e_i \in \text{int} X_i\), then

\[
B_i(p, q, \mathcal{F}_\pi) = \text{cl}(x, y) \in X_i \times \pi Z_i : p \square (x - e_i) \ll W(p, q)y_i.
\]

We show that if the economy \((\mathcal{E}, \mathcal{F})\) has an equilibrium \((p^*, x^*, q^*, z^*)\), then \((p^*, x^*, \pi q^*, \pi z^*)\) is an equilibrium of \((\mathcal{E}, \mathcal{F}_\pi)\). Let \((p^*, x^*, q^*, z^*)\) be an equilibrium in \((\mathcal{E}, \mathcal{F})\). The asset market clearing condition in \((\mathcal{E}, \mathcal{F}_\pi)\): \(\sum_i \pi z^*_i = 0\) is a direct consequence of \(\sum_i z^*_i = 0\). First, we show that for each \(i \in \mathbf{I}\), \((x^*_i, \pi z^*_i) \in B_i^{\text{cl}}(p^*, \pi q^*)\). It suffices to show that \(W(p^*, \pi q^*)z^*_i = W(p^*, q^*)z^*_i\). From (4.1) we have \(W(p^*, \pi q^*)z^*_i = W(p^*, q^*)z^*_i\). Recall that by Proposition 1, \((q^*, z^*)\) is arbitrage-free at \(p^*\) in \(\mathcal{F}\). Thus by Lemma 3, \(q^* \in \mathcal{L}^+_T\). Hence \(q^* \in (\ker \pi)^+_i\) since \(\ker \pi \subset \mathcal{L}_T\). Therefore \(q^* \cdot \pi z^*_i = q^* \cdot z^*_i\) and \(W(p^*, q^*)z^*_i = W(p^*, q^*)z^*_i\), recalling that \(V(p^*)z^*_i = V(p^*)z^*_i\) from (4.1). We now show that for each \(i \in \mathbf{I}\), \((x^*_i, \pi z^*_i)\) solves agent \(i\)'s problem in \((\mathcal{E}, \mathcal{F}_\pi)\). Suppose on the contrary that for some agent, say \(i = 1\), there exists \((x_1, z_1) \in B_1^{\text{cl}}(p^*, \pi q^*)\) such that \(x_1 \in P_1(x^*)\). Recall that by \(\text{LNS}\) one
has $p^*(s) \neq 0$ for all $s \in S$. From Lemma 4, \((x_i, z_i) = \lim_n (x_i^n, \pi z_i^n)\) for some sequences \((x_i^n) \subset X_i\) and \((z_i^n) \subset Z_i\) such that
\[
p^* \sqcap (x_i^n - e_1) - W(p^*, \pi q^*) (\pi z_i^n) \leq 0.
\]
We have $W(p^*, \pi q^*) (\pi z_i^n) = W(p^*, q^*) (\pi z_i^n) = W(p^*, q^*) (\pi z_i^n)$ (the first equality comes from (4.1) and the second equality is a consequence of the fact that under Assumption FA, $q^* \in L^+_F$ by Lemma 3 and therefore $q^* \in (\ker \pi)^\perp$ since $\ker \pi \subset L^+_F$). Consequently (from above),
\[
p^* \sqcap (x_i^n - e_1) - W(p^*, q^*) (\pi z_i^n) = p^* \sqcap (x_i^n - e_1) - W(p^*, \pi q^*) (\pi z_i^n) \leq 0.
\]
Hence \((x_i^n, z_i^n) \in B_1 (p^*, q^*, F)\). Recalling that $x_1 \in P_1(x^*)$, $x_1 = \lim_n x_i^n$ and using the fact that $P_1(x^*)$ is open, we deduce that for $n$ large enough $x_i^n \in P_1(x^*)$. The two assertions \((x_i^n, z_i^n) \in B_1 (p^*, q^*, F)\) and \(x_1 \in P_1(x^*)\) contradict the optimality of \((x_i^n, z_i^n)\) in \((E, F)\).

5. Appendix

5.1. Reducibility, WNMA, and Closedness

**Proposition 4.** Let $F$ be a financial structure.

(a) If $F$ is reduced then it satisfies WNMA.

(b) If $F$ satisfies F2 then it is closed.

(c) The converses in (a) and (b) above do not hold true.

(d) For $p \in \mathbb{R}^L$, denote
\[
\mathcal{G}_F(p) := \{(v_i), \sum_{i \in I} z_i \in (\mathbb{R}^S)_I \times \mathbb{R}^J : \forall i \in I, z_i \in Z_i, V(p)z_i \geq v_i\}.
\]

Then the set $\mathcal{G}_F(p)$ is closed if and only if the set $\mathcal{G}_F(p)$ is closed.

**Proof.** (a) Let $p \in \mathbb{R}^L$ and let \((\zeta_1, \ldots, \zeta_l) \in \prod(AZ_i \cap \{V(p) \geq 0\})\) be such that $\sum_{i \in I} \zeta_i = 0$. Then \((\zeta_1, \ldots, \zeta_l) \in A(A_F(v))$. Since $A_F(v)$ is bounded, one has $A(A_F(v)) = \{0\}$ (see Rockafellar (1997)). Hence, for all $i$, $\zeta_i = 0 \in AZ_i \cap -AZ_i \cap \ker V(p)$.

(b) F2(i) $\Rightarrow$ Closedness: First, we prove the result when for every $i \in I$, $K_i = \{0\}$, i.e. when $Z_i$ is polyhedral for every $i$. Let
\[
f : \mathbb{R}^J \rightarrow \mathbb{R}^S \times \mathbb{R}^J, (z_1, \ldots, z_i) \mapsto (V(p)z_1, \ldots, V(p)z_i, \sum_{i \in I} z_i).
\]
Then $f$ is linear and one has $\mathcal{G}_F(p) = f(\prod I_i Z_i)$. Since $\prod I_i Z_i$ is polyhedral, Theorem 19.3 page 174 in Rockafellar (1997) allows to conclude that $\mathcal{G}_F(p)$ is polyhedral, hence closed.

Now, we show the result in the general case. Let \((V(p)z_1^n, \ldots, V(p)z_i^n, \sum_{i \in I} z_i^n)\) be a sequence in the set $\mathcal{G}_F(p)$ such that \((V(p)z_1^n, \ldots, V(p)z_i^n, \sum_{i \in I} z_i^n) \rightarrow (v_1, \ldots, v_i, y)\), where \((z_i^n) \subset Z_i\) for every $i \in I$. By assumption, for all $i$ and for all $n$, $z_i^n = k_i^n + \beta_i^n$ where $k_i^n \in K_i$ and $\beta_i^n \in P_i$. Since the $K_i$'s are compact, we can assume $k_i^n \rightarrow k_i \in K_i$ for all $i \in I$. The sequence \((V(p)z_i^n - V(p)k_i^n), \sum_{i \in I} z_i^n - k_i^n)\) is
\( \sum_{i \in I} k_i^n \) is in the set \( \mathcal{G}_{F'}(p) \), where \( F' = (V, (P_i)) \). Since, by the first part of the proof, \( \mathcal{G}_{F'}(p) \) is closed, one has

\[
\left( (v_i - V(p)k_i), y - \sum_{i \in I} k_i \right) = \lim_n \left( (V(p)z_i^n - V(p)k_i), \sum_{i \in I} z_i^n - \sum_{i \in I} k_i \right) \in \mathcal{G}_{F'}(p). 
\]

Hence, for all \( i \) there exists \( \beta_i \in P_i \) such that \( V(p)\beta_i = v_i - V(p)k_i \) and \( y - \sum_{i \in I} k_i = \sum_{i \in I} \beta_i \). Therefore \( \left( (v_i), y \right) = \left( (V(p)(k_i + \beta_i)), \sum_{i \in I}(k_i + \beta_i) \right) \) with \( k_i + \beta_i \in Z_i \) for each \( i \). That is, \( \left( (v_i), y \right) \in \mathcal{G}_{F}(p) \).

**F2(ii) ⇒ Closedness:** We show that if the sets \( AZ_i \cap \ker V(p) \) are WPSI then the set \( \mathcal{G}_{F}(p) \) is closed. We have

\[
\mathcal{G}_{F}(p) = \{ (V(p)z_1, \ldots, V(p)z_I, \sum_{i \in I} z_i) : \forall i, z_i \in Z_i \} = \sum_{i \in I} X_i
\]

with

\[
X_i = \{ (0, \ldots, 0, V(p)z_i, 0, \ldots, 0, z_j) : z_j \in Z_j \}.
\]

Then \( AX_i = \{ (0, \ldots, 0, V(p)\zeta, 0, \ldots, 0, \zeta) : \zeta \in AZ_i \} \). Now we show that the sets \( AX_i \) (\( i \in I \)) are weakly positively semi-independent. This will end the proof (see Theorem 9.1 page 73 in Rockafellar (1997)). If \( \sum_{i \in I} w_i = \sum_{i \in I} (0, \ldots, 0, V(p)\zeta, 0, \ldots, 0, \zeta) = 0 \) with \( \zeta \in AZ_i \), then for every \( i \), \( V(p)\zeta = 0, \zeta \in AZ_i \), and \( \sum_{i \in I} \zeta_i = 0 \). Hence for each \( i \), \( \zeta_i \in AZ_i \cap \ker V(p) \) and \( \sum_{i \in I} \zeta_i = 0 \). By WPSI of the sets \( AZ_i \cap \ker V(p) \), we get \( \zeta_i \in AZ_i \cap -AZ_i \) for each \( i \). Hence \( w_i \in AX_i \cap -AX_i \) for each \( i \in I \).

(c) • An example where \( F \) satisfies F2(i) and is not reduced: Let \( V = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \), \( I = 2 \), and

\[
Z_1 = \{ (z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 \geq 0, z_2 \in \mathbb{R}, z_3 \in \mathbb{R} \},
\]

\[
Z_2 = Z_1.
\]

The \( Z_i \)’s are polyhedral convex, hence \( F \) satisfies F2(i). It is easy to check that \( \{ V \geq 0 \} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \), \( AZ_1 = AZ_2 = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \). Thus \( AZ_1 \cap \{ V \geq 0 \} = AZ_2 \cap \{ V \geq 0 \} = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \). So \( AZ_i \cap -AZ_i \cap \ker V = \{ 0 \} \times \mathbb{R} \times \{ 0 \} \). So the cones \( AZ_i \cap \{ V \geq 0 \} \) are not PSI (hence \( F \) is not reduced).

• An example where \( F \) satisfies F2(ii) and is not reduced: Let \( V = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \), \( I = 2 \), and

\[
Z_1 = \{ (z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 \in \mathbb{R}, z_2 \in \mathbb{R}, z_3 \geq z_1^2 \},
\]

\[
Z_2 = Z_1.
\]

It is easy to check that \( \{ V \geq 0 \} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \), \( AZ_1 = AZ_2 = \{ 0 \} \times \mathbb{R} \times \mathbb{R}_+ \). Thus \( AZ_i \cap \{ V \geq 0 \} = \{ 0 \} \times \mathbb{R} \times \mathbb{R}_+ \). So the cones \( AZ_i \cap \{ V \geq 0 \} \) are WPSI but not PSI (hence \( F \) is not reduced).

• An example where \( F \) is closed and does not satisfy F2: Consider two agents, two states of nature at the second dare, and three assets i.e. \( I = S = 2 \), and \( J = 3 \). Let the payoff matrix be given by

\[
V = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.
\]
Let agents’ portfolio sets be given by (note that $Z_1$ is not polyhedral)
\[
Z_1 = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 \geq 0, z_2 \geq 0, z_3^2 \leq (z_1 + 1)z_2\},
\]
\[
Z_2 = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 \geq 0, z_2 \leq 0, z_3 = 0\}.
\]

We show that the collection \(\{AZ_i \cap \{V \geq 0\} : i \in 1, 2\}\) is not WPSI and \(\mathcal{F} = \left\{(V, (Z_i))_{i=1,2}\right\}\) is closed. Clearly \(AZ_2 = Z_2\) and it is easy to check that \(\)
\[
AZ_1 = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 \geq 0, z_2 \geq 0, z_3^2 \leq z_1 z_2\},
\]

and \(\{V \geq 0\} = \{0\} \times \mathbb{R} \times \mathbb{R}\). Hence \(AZ_1 \cap \{V \geq 0\} = \{0\} \times \mathbb{R} \times \{0\}\), and \(AZ_2 \cap \{V \geq 0\} = \{0\} \times \mathbb{R} \times \{0\}\). Then the collection \(\{AZ_i \cap \{V \geq 0\} : i \in 1, 2\}\) is not WPSI. We show that \(\mathcal{F} = \left\{(V, (Z_i))_{i=1,2}\right\}\) is closed. Let \((z^n_i)_n\) be a sequence in \(Z_i (i = 1, 2)\), and assume that \((V z^n_1, V z^n_2, z^n_1 + z^n_2) \rightarrow_{n \rightarrow \infty} (v_1, v_2, z)\). We need to show that there exist \((z_1, z_2) \in Z_1 \times Z_2\) such that \((v_1, v_2, z) = (V z_1, V z_2, z_1 + z_2)\). Write \(z^n_1 = (\alpha^n_1, \beta^n_1, \gamma^n_1)\) and \(z^n_2 = (\alpha^n_2, \beta^n_2, \gamma^n_2)\) with
\[
\begin{align*}
\alpha^n_1 &\geq 0, \quad \beta^n_1 \geq 0, \quad (\gamma^n_1)^2 \leq (\alpha^n_1 + 1)\beta^n_1, \\
\alpha^n_2 &\geq 0, \quad \beta^n_2 \leq 0, \quad \gamma^n_2 = 0.
\end{align*}
\]

Then \(V z^n_1 = (\alpha^n_1, -\alpha^n_1), V z^n_2 = (\alpha^n_2, -\alpha^n_2),\) and \(z^n_1 + z^n_2 = (\alpha^n_1 + \alpha^n_2, \beta^n_1 + \beta^n_2, \gamma^n_1 + \gamma^n_2)\). Since the sequences \((V z^n_i)_{n} (i = 1, 2)\) converge, we have \(\alpha^n_1 \rightarrow_{n \rightarrow \infty} \alpha_1 \geq 0, \alpha^n_2 \rightarrow_{n \rightarrow \infty} \alpha_2 \geq 0,\) and from the convergence of the sequence \((\gamma^n_1 + \gamma^n_2)_{n}\), and the fact that \(\gamma^n_2 = 0\) we conclude that \(\gamma^n_1 \rightarrow_{n \rightarrow \infty} \gamma_1\). Denote \(s = \lim_n(\beta^n_1 + \beta^n_2)\). Choose \(\beta_1 \geq \max\left(s, (\gamma_1)^2/(\alpha_1 + 1)\right)\), and let \(\beta_2 = s - \beta_1\). Then one can easily check that \(z_1 := (\alpha_1, \beta_1, \gamma_1) \in Z_1, z_2 := (\alpha_2, \beta_2, 0) \in Z_2,\) and \(v_1 = V z_1, v_2 = V z_2, z = z_1 + z_2\).

(d) Assume \(\mathcal{G}_\mathcal{F}(p)\) is closed and let \((w^n)_n\) be a sequence in \(\mathcal{G}_\mathcal{F}(p)\) which converges to some \(w \in (\mathbb{R}^S)^I \times \mathbb{R}^I\) i.e. \(w^n = (V(p)z^n_i, \cdots, V(p)z^n_I, \sum_{i \in I}z^n_i) \rightarrow_{n \rightarrow \infty} w = (v_1, \cdots, v_I, z),\) with \(z^n_i \in Z_i\) for each \(i \in I\) and for every \(n \in \mathbb{N}\). Then \(w^n \in \mathcal{G}_\mathcal{F}(p)\) for every \(n\), and since \(\mathcal{G}_\mathcal{F}(p)\) is closed, we have \(w \in \mathcal{G}_\mathcal{F}(p)\). That is \(z = \sum_{i \in I}z_i\) with \(z_i \in Z_i\) and \(V(p)z_i \geq v_i\) for every \(i \in I\). But \(\sum_{i \in I}v_i = \sum_{i \in I}\lim_n V(p)z^n_i = \lim_n V(p)(\sum_{i \in I}z^n_i) = V(p)z = V(p)(\sum_{i \in I}z_i)\), hence \(v_i = V(p)z_i\) for each \(i \in I\), and consequently, \(w = (V(p)z_1, \cdots, V(p)z_I, \sum_{i \in I}z_i) \in \mathcal{G}_\mathcal{F}(p)\).

Conversely, assume \(\mathcal{G}_\mathcal{F}(p)\) closed and let \((w^n)_n\) be a sequence in \(\mathcal{G}_\mathcal{F}(p)\) which converges to some \(w' \in (\mathbb{R}^S)^I \times \mathbb{R}^I\) i.e. \(w^n = (v^n_1, \cdots, v^n_I, \sum_{i \in I}z^n_i) \rightarrow_{n \rightarrow \infty} w' = (v'_1, \cdots, v'_I, z),\) with \(z^n_i \in Z_i\) and \(V(p)z^n_i \geq v'_i\) for each \(i \in I\) and for every \(n \in \mathbb{N}\). For each \(i \in I\), the sequence \((v^n_i)_n\) converges hence is bounded, therefore the sequence \((V(p)z^n_i)_n\) is bounded below (since \(V(p)z^n_i \geq v'_i\) for every \(n\)). Moreover the sequence \((\sum_{i \in I}V(p)z^n_i)_n\) converges (towards \(V(p)z\)), hence for each \(i \in I,\) the sequence \((V(p)z^n_i)_n\) is bounded. We can therefore assume that for each \(i \in I,\) the sequence \((V(p)z^n_i)_n\) converges (use subsequences if necessary) to \(v_i \in \mathbb{R}^S\) satisfying \(v_i \geq v'_i\). Now we consider the sequence \((w^n)_n \in \mathcal{G}_\mathcal{F}(p)\) where \(w^n = (V(p)z^n_1, \cdots, V(p)z^n_I, \sum_{i \in I}z^n_i)\). Then from above, \(w^n \rightarrow_{n \rightarrow \infty} w = (v_1, \cdots, v_I, z) \in \mathcal{G}_\mathcal{F}(p)\) (since \(\mathcal{G}_\mathcal{F}(p)\) is assumed to be closed). Hence \(z\) can be written as \(z = \sum_{i \in I}z_i\) with \(z_i \in Z_i\) and \(V(p)z_i = v_i\) for each \(i \in I\). Recall that \(V(p)z_i = v_i \geq v'_i\) for each \(i \in I\) and that \(w' = w' = (v'_1, \cdots, v'_I, z) = (v'_1, \cdots, v'_I, \sum_{i \in I}z_i)\), hence \(w' \in \mathcal{G}_\mathcal{F}(p)\).
5.2. Proof of Lemma 1

We prepare the proof by two claims.

Claim 5.1. Let $\mathcal{F} = (V_i, (Z_i))$ be a standard financial structure. If $\mathcal{F}$ is closed then for every $v = (v_i)_{i \in I} \in (\mathbb{R}^S)^I$ such that $\sum_{i \in I} (Z_i \cap \{V(p) \geq v_i\}) \neq \emptyset$, the space $L_{\mathcal{F}}(p, v) := A\left(\sum_{i \in I} (Z_i \cap \{V(p) \geq v_i\})\right)$ does not depend on $v$.

**Proof.** We show that for all $p \in \mathbb{R}^L$ and for all $v = (v_i)_{i \in I} \in (\mathbb{R}^S)^I$ such that $\sum_{i \in I} (Z_i \cap \{V(p) \geq v_i\})$ and $\sum_{i \in I} (Z_i \cap \{V(p) \geq w_i\})$ are not empty, we have

$$\mathcal{A}\left(\sum_{i \in I} (Z_i \cap \{V(p) \geq v_i\})\right) \subset \mathcal{A}\left(\sum_{i \in I} (Z_i \cap \{V(p) \geq w_i\})\right).$$

The result of the proposition is obviously a direct consequence of the above inclusion.

Let $\zeta \in \mathcal{A}\left(\sum_{i \in I} (Z_i \cap \{V(p) \geq v_i\})\right)$, then $\zeta = \lim_{n \to \infty} \lambda_n \sum_{i \in I} z_i^n$ for some $z_i^n \in Z_i \cap \{V(p) \geq v_i\}$, $\lambda_n > 0$, and $\lambda_n \downarrow 0$. We need to show that $\zeta \in \mathcal{A}\left(\sum_{i \in I} (Z_i \cap \{V(p) \geq w_i\})\right)$, that is, for $z_i \in Z_i$ ($i \in I$) such that $V(p)z_i \geq w_i$, we have

$$\zeta + \sum_{i \in I} z_i \in \sum_{i \in I} (Z_i \cap \{V(p) \geq w_i\}).$$

From above,

$$\zeta + \sum_{i \in I} z_i = \lim_{n \to \infty} \sum_{i \in I} \left(\lambda_n z_i^n + (1 - \lambda_n)z_i\right).$$

Notice that, for $n$ large enough, $\lambda_n \in [0, 1]$. Hence $y_i^n := \lambda_n z_i^n + (1 - \lambda_n)z_i$ belongs to $Z_i$ (because $z_i^n$ and $z_i$ are in $Z_i$, and $Z_i$ is convex). Furthermore $V(p)y_i^n \geq \lambda_n v_i + (1 - \lambda_n)w_i$. Therefore the sequence $(V(p)y_i^n)_n$ is bounded. Moreover the sequence $(\sum_{i \in I} (V(p)y_i^n)_n$ converges (towards $(\sum_{i \in I} \zeta_i)$), hence for each $i \in I$, the sequence $(V(p)y_i^n)_n$ is bounded. We can henceforth assume that for each $i \in I$, the sequence $(V(p)y_i^n)_n$ converges (use subsequences if necessary) to $v_i' \in \mathbb{R}^S$ satisfying $v_i' \geq w_i$. Since

$$\left(V(p)y_i^n, \ldots, V(p)y_i^n, \sum_{i \in I} y_i^n\right) \in \mathcal{G}_{\mathcal{F}}(p),$$

$$\left(V(p)y_i^n, \ldots, V(p)y_i^n, \sum_{i \in I} y_i^n\right) \to (v_i', \ldots, v_i', \xi + \sum_{i \in I} z_i),$$

and the set $\mathcal{G}_{\mathcal{F}}(p)$ is closed (by the Closedness Assumption), we conclude that there exists $(y_i)_{i \in I} \in \prod_i Z_i$ such that $(v_i', \ldots, v_i', \xi + \sum_{i \in I} z_i) = (V(p)y_1, \ldots, V(p)y_i, \sum_{i \in I} y_i)$. Hence, recalling that from above, $v_i' \geq w_i$, we have $\zeta + \sum_{i \in I} z_i = \sum_{i \in I} y_i \in \sum_{i \in I} Z_i \cap \{V(p) \geq w_i\}$. 

**Claim 5.2.** Let $\mathcal{F} = (V_i, (Z_i))$ be standard and closed. For all $p \in \mathbb{R}^L$ and for all $(v_i)_{i \in I} \in (\mathbb{R}^S)^I$, one has

$$\sum_{i \in I} (\pi Z_i \cap \{V(p) \geq v_i\}) \subset \text{cl} \sum_{i \in I} (Z_i \cap \{V(p) \geq v_i\}).$$
Proof. If \( \sum_{i \in I} (\pi Z_i \cap \{ V(p) \geq v_i \}) = \emptyset \) then there is nothing to prove. Otherwise, we show that
\[
\sum_{i \in I} (\pi Z_i \cap \{ V(p) \geq v_i \}) \subset \sum_{i \in I} \pi (Z_i \cap \{ V(p) \geq v_i \}) \quad (5.1)
\]
\[
\subset \ker \pi + \sum_{i \in I} (Z_i \cap \{ V(p) \geq v_i \}) \quad (5.2)
\]
\[
\subset \left( \sum_{i \in I} (Z_i \cap \{ V(p) \geq v_i \}) \right). \quad (5.3)
\]

To prove the inclusion (5.1), it suffices to notice that for every \( i \in I, \pi Z_i \cap \{ V(p) \geq v_i \} \subset \pi (Z_i \cap \{ V(p) \geq v_i \}) \). Indeed, let \( y_i \in \pi Z_i \cap \{ V(p) \geq v_i \} \), then there exists \( z_i \in Z_i \) such that \( y_i = \pi z_i \), and \( V(p) y_i \geq v_i \). But \( V(p) z_i = V(p) y_i + V(p) (z_i - \pi z_i) = V(p) y_i \) (since \( z_i - \pi z_i \in \ker \pi \subset L_\pi \) and obviously \( L_\pi \subset \ker V(p) \)). Then \( z_i \in Z_i \cap \{ V(p) \geq v_i \} \) and consequently \( y_i \in \pi (Z_i \cap \{ V(p) \geq v_i \}) \).

To prove the inclusion (5.2), let \( y = \sum_{i \in I} \pi z_i \) with \( z_i \in Z_i \cap \{ V(p) \geq v_i \} \). Then \( y = \pi z = (\pi z - z) + z \) with \( \pi z - z \in \ker \pi \) and \( z = \sum_{i \in I} z_i \in \sum_{i \in I} (Z_i \cap \{ V(p) \geq v_i \}) \). This ends the proof of the inclusion of (5.2).

The last inclusion (5.3) comes from the fact that
\[
\ker \pi \subset L_\pi = A \left( \sum_{i \in I} (Z_i \cap \{ V(p) \geq v_i \}) \right) \cap -A \left( \sum_{i \in I} (Z_i \cap \{ V(p) \geq v_i \}) \right) \subset A \left( \sum_{i \in I} (Z_i \cap \{ V(p) \geq v_i \}) \right),
\]
where the first inclusion holds by assumption, the equality comes from Claim 5.1, and the last inclusion is immediate. Consequently,
\[
\ker \pi + \sum_{i \in I} (Z_i \cap \{ V(p) \geq v_i \}) \subset A \left( \sum_{i \in I} (Z_i \cap \{ V(p) \geq v_i \}) \right) + \sum_{i \in I} (Z_i \cap \{ V(p) \geq v_i \})
\]
\[
\subset \text{cl} \left( \sum_{i \in I} (Z_i \cap \{ V(p) \geq v_i \}) \right) = \sum_{i \in I} (Z_i \cap \{ V(p) \geq v_i \}).
\]
The last equality follows from the closedness assumption. This ends the proof of the claim.

Proof of Lemma 1

Let \( y_i \in (\text{cl} \pi Z_i) \cap \{ V(p) \geq V(p) \hat{y}_i \} \) (\( i \in I \)). Take \( v_i^n \uparrow V(p) \hat{y}_i \) such that \( V(p) \hat{y}_i \geq v_i^n \) for every \( n \). Pick \( \tilde{y}_i \in r \pi Z_i \) and consider \( y_i^n = (1 - x^n) y_i + x^n \tilde{y}_i \) with \( 0 < x^n < \frac{1}{n} \) small enough so that \( V(p) y_i^n \geq v_i^n \). Then \( y_i^n \in [\tilde{y}_i, y_i] \subset r \pi Z_i \) since \( y_i \in \text{cl} \pi Z_i \) and \( \tilde{y}_i \in r \pi Z_i \) (Theorem 6.1 page 45 in Rockafellar (1997)). Thus \( y_i^n \in \pi Z_i \cap \{ V(p) \geq v_i^n \} \). Therefore, by Claim 5.2,
\[
\sum_{i \in I} y_i^n \in \left( \sum_{i \in I} (Z_i \cap \{ V(p) \geq v_i^n \}) \right).
\]
Since \( G_p' (p) := \{(v_i), \sum_{i \in I} z_i \} \in (\mathbb{R}^s)^I \times \mathbb{R}^I : \forall i \in I, z_i \in Z_i, V(p) z_i \geq v_i \} \) is closed by Proposition 4,
\[
(V(p) \hat{y}_1, \ldots, V(p) \hat{y}_I, \sum_{i \in I} y_i) = \lim_n (v_i^n, \ldots, v_i^n, \sum_{i \in I} z_i^n) \in \text{cl} G_p' (p) = G_p' (p).
\]
Hence \( \sum_{i \in I} y_i \in \left( \sum_{i \in I} (Z_i \cap \{ V(p) \geq V(p) \hat{y}_i \}) \right). \) This completes the proof of the first part of the lemma. Taking \( \hat{y}_i = 0 \) for each \( i \in I \), one has \( \sum_{i \in I} ((\text{cl} \pi Z_i) \cap \{ V(p) \geq 0 \}) \subset \text{cl} \sum_{i \in I} (Z_i \cap \{ V(p) \geq 0 \}) \). Hence, \( L_\pi \subset A \left( \text{cl} \sum_{i \in I} Z_i \cap \{ V(p) \geq 0 \} \right) \cap -A \left( \text{cl} \sum_{i \in I} Z_i \cap \{ V(p) \geq 0 \} \right) = L_\pi \). This ends the proof of the lemma.

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5.3. **Proof of Lemma 2**

To show that the closed set $\mathcal{A}_F(p, v)$ is compact, it suffices to show that $A(\mathcal{A}_F(p, v)) = \{0\}$ (see Rockafellar (1997)). We have

$$A(\mathcal{A}_F(p, v)) = \{(\zeta_1, \ldots, \zeta_i) \in \prod_{i \in I} AZ_i : \forall i, V(p)\zeta_i \geq 0, \sum_{i \in I} \zeta_i = 0\}.$$  

Let $(\zeta_1, \ldots, \zeta_i) \in A(\mathcal{A}_F(p, v))$. Then

$$\zeta_i = -\sum_{k \neq i} \zeta_k \in (AZ_i \cap \{V(p) \geq 0\}) \cap \left(\sum_{k \neq i} (AZ_k \cap \{V(p) \geq 0\})\right) \subseteq A_F \cap -A_F = \{0\}.$$  

Hence $\zeta_i = 0$ for every $i$. Therefore $(\zeta_1, \ldots, \zeta_i) = (0, \ldots, 0)$.

5.4. **Proof of Lemma 3**

By contraposition. Let $(\vec{q}, \vec{z})$ be arbitrage-free at $\vec{p}$ in $\mathcal{F}$ and suppose that $\vec{q} \notin A\left(\sum_{i \in I}(Z_i \cap \{V(p) \geq 0\})\right)$. Then there exists $\zeta \in A\left(\sum_{i \in I}(Z_i \cap \{V(p) \geq 0\})\right)$ such that $-\vec{q} \cdot \zeta > 0$. Thus, for every $n \in \mathbb{N}$, $n^2 \zeta = \sum_{i \in I} z^n_i$ for some $z^n_i \in Z_i \cap \{V(p) \geq 0\}$. Therefore $-\vec{q} \cdot \sum_{i \in I}(z^n_i / n) = -n\vec{q} \cdot \zeta \to +\infty$. Hence, without any loss of generality, one can assume that for some agent, say $i = 1$, $-\vec{q} \cdot (z^n_1 / n) \to +\infty$.

By FA, there exists $\vec{\epsilon}_1 \in AZ_1$ such that $V(\vec{p})\vec{\epsilon}_1 \gg 0$. Define

$$\vec{\epsilon}_1 := \frac{1}{n} z^n_1 + (1 - \frac{1}{n})(\vec{z}_1 + \vec{\epsilon}_1).$$

We end the proof by showing that (i) $z^n_1 \in Z_1$, and (ii) for $n$ large enough, $W(\vec{p}, \vec{q})z^n_1 \gg W(\vec{p}, \vec{q})\vec{\epsilon}_1$. 

In other words, we show that $z^n_1$ is an arbitrage opportunity for agent 1 at $\vec{z}_1$ in $\mathcal{F}$, which is a contradiction to the fact that $(\vec{q}, \vec{z})$ is arbitrage-free at $\vec{p}$ in $\mathcal{F}$. First, since $\vec{\epsilon}_1 \in AZ_1$, one has $\vec{z}_1 + \vec{\epsilon}_1 \in Z_1$, and since $z^n_1 \in Z_1$ and $\vec{z}_1 + \vec{\epsilon}_1 \in Z_1$, the convexity of $Z_1$ (since $\mathcal{F}$ is standard) allows to conclude that $z^n_1$ belongs to $Z_1$. Second, since $-\vec{q} \cdot (z^n_1 / n) \to +\infty$, one has, for $n$ large enough $-\vec{q} \cdot z^n_1 = -\vec{q} \cdot \frac{1}{n} z^n_1 + -\vec{q} \cdot (1 - \frac{1}{n})(\vec{z}_1 + \vec{\epsilon}_1) > -\vec{q} \cdot \vec{z}_1$.

Finally, since $z^n_1 \in \{V(\vec{p}) \geq 0\}$ and $V(\vec{p})\vec{\epsilon}_1 \gg 0$, one has, for $n$ large enough

$$V(\vec{p})z^n_1 = V(\vec{p})\left(\frac{1}{n} z^n_1 + (1 - \frac{1}{n})(\vec{z}_1 + \vec{\epsilon}_1)\right) \geq (1 - \frac{1}{n})V(\vec{p})(\vec{z}_1 + \vec{\epsilon}_1) \gg V(\vec{p})\vec{\epsilon}_1.$$ 

Hence, for $n$ large enough, $W(\vec{p}, \vec{q})z^n_1 > W(\vec{p}, \vec{q})\vec{\epsilon}_1$. This ends the proof of the claim.

5.5. **Proof of Lemma 4**

We first claim that there exists $\delta = (\delta(s))_{s \in S} \in \mathbb{R}^L$ such that (i) $e_i - \delta \in X_i$ and (ii) $p(s) \cdot \delta(s) > 0$ for every $s \in S$. Indeed, take $\delta = \lambda p$ for $\lambda > 0$ small enough so that $e_i - \delta \in X_i$ using the fact that $e_i \in \text{int}X_i$. Then, for all $s \in S$, $p(s) \cdot \delta(s) = \lambda p(s) \cdot p(s) > 0$, since $p(s) \neq 0$. Let $(x_i, y_i) \in B^*_{F_s}(p, q)$. Let $\alpha \in (0, 1)$. Then $x_i^\alpha := \alpha x_i + (1 - \alpha)(e_i - \delta) \in X_i$ since $x_i \in X_i, e_i - \delta \in X_i$ and $X_i$ is convex, and $\alpha y_i \in \text{cl}\pi Z_i$ since $0 \in \text{cl}\pi Z_i, y_i \in \text{cl}\pi Z_i$, and $\text{cl}\pi Z_i$ is convex. We claim that,

$$p \cdot (x_i^\alpha - e_i) - W(p, q)(\alpha y_i) \ll 0.$$
Indeed, $p \square (x^\alpha_i - e_i) - W(p, q)(\alpha y_i) = \alpha \left( p \square (x_i - e_i) - W(p, q)y_i \right) - (1 - \alpha)p \square \delta$. Since $(x_i, y_i) \in B^\mathcal{F}_\mathcal{P}(p, q)$, i.e., $p \square (x_i - e_i) - W(p, q)y_i \leq 0$, and $\alpha > 0$, the first term is nonpositive. Since $p \square \delta \gg 0$ (from above) and $\alpha < 1$, the second term satisfies $-(1 - \alpha)p \square \delta \ll 0$. This ends the proof of the claim. Consequently, there exists $y^\alpha_i \in \pi Z_i$ such that $\|y^\alpha_i - y_i\| \leq (1 - \alpha)\|y_i\|$ and

$$p \square (x^\alpha_i - e_i) - W(p, q)y^\alpha_i \ll 0.$$  

Noticing that, $(x^\alpha_i, y^\alpha_i) \rightarrow (x_i, y_i)$ when $\alpha \rightarrow 1$, we get the desired result. 

References


