Share Equilibrium in Local Public Good Economies*

by

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Abstract We define a concept of share equilibrium for local public good (or club) economies where players may have preferences over the membership of their jurisdiction. A share equilibrium specifies one share index for each player in an economy. These indices determine each player’s cost shares in any jurisdiction that he might join. We demonstrate that the same axioms as those that characterize the Lindahl equilibrium, as discussed in his 1919 paper, also characterize the share equilibrium. In addition to our axiomatization, we establish that share equilibrium outcomes are in the core. We provide an existence of equilibrium result for a special case and we also relate our model to well-known models of economies with local public goods.
1 An introduction to share equilibrium

This paper introduces the concept of a “share equilibrium” in the context of an economy with local public goods. A share equilibrium endogenously determines share indices for all players. A player’s relative share index determines the player’s share of the costs in any jurisdiction to which he might belong. In many situations involving clubs, local public goods, or group activities, the shares of the costs of group activities are determined by some relative share indices of the individuals involved in the activity. Two examples that may fit very well are: property taxes that depend on property evaluations and thus on relative property values and; condominium homeowners fees that depend on relative sizes of condominium units. We provide parametric examples that illustrate share equilibrium for an economy with condominiums.

Part of our motivation for the share equilibrium is to provide an extension of the ratio equilibrium concept, introduced by Kaneko (1977a, 1977b), to economies with local public goods. The ratio equilibrium is an appealing concept and well reflects the ideas of cost sharing in Lindahl (1919). While with constant returns to scale in production of (pure) public goods, the Lindahl equilibrium, as described in Lindahl’s original paper, yields the same outcomes as the equilibrium concept defined in Samuelson (1954), the two equilibrium concepts have different underlying philosophies and are fundamentally different. This becomes clear when considering properties of the concepts. In van den Nouweland, Tijs and Wooders (2002) we translate the ideas of Lindahl into axioms and find a set of axioms that are satisfied by the ratio equilibrium but not by the Samuelson equilibrium. In the current paper we identify an equilibrium for local public good economies, the share equilibrium, that satisfies axioms analogous to those in our axiomatic char-
acterization of the ratio equilibrium.\textsuperscript{1} In the remainder of this introduction, we provide a more detailed description of the share equilibrium, the model, and our results. Since a share equilibrium is a new concept, there are many open questions; in a concluding section, we report on research in progress dealing with some of these.

Recall that a (pure) public good is a commodity that can be consumed in its entirety by all players in an economy. Thus, cost-sharing rules for public goods cannot be determined by competition between players for the available supplies of the commodity. Various solutions to the problem of allocation of costs of pure public good provision have been proposed. The most well-known is perhaps the Lindahl equilibrium, as formalized in Samuelson (1954), Johansen (1963), and Foley (1970), with personalized prices for public goods.\textsuperscript{2} In contrast, however, according to our reading of Lindahl’s (1919) paper (see van den Nouweland, Tijs and Wooders 2002), players pay shares of the total costs and, in equilibrium, these shares must satisfy the property that the amount of public good demanded is the same for all individuals, given their cost shares. Other papers that have taken approaches in a similar spirit, with individuals paying shares of costs rather than per unit prices, include Kaneko (1977a,b) and Mas-Colell and Silvestre (1989,1991).

In an economy with local public goods as suggested by Tiebout (1956), public goods are subject to exclusion and possibly also to congestion and a state of the economy includes a partition of the set of players into jurisdictions; the members of a jurisdiction provide the public goods exclusively for

\textsuperscript{1}Axiomatic characterization of solution concepts is a well-established approach to their study. Recent contributions include, for example, Moulin (2000), Dhillon and Mertens (1999), and Maskin (1999).

\textsuperscript{2}See also Mas-Colell (1980) where the concept of public goods was extended to public projects.
the use of the jurisdiction membership.\textsuperscript{3} Thus, in a local public good economy an equilibrium must determine the cost share of each player for each jurisdiction he might join. In a share equilibrium, this is accomplished by means of share indices. The specification of a share equilibrium includes a state of the economy and a share index for each player. The cost share of a player in any jurisdiction to which he might belong is given by his share index divided by the sum of the share indices of all the players in the jurisdiction. We emphasize that share indices reflect relative cost shares of players in various jurisdictions and thus a share equilibrium depends on only one set of indices — each player $i$ is assigned a single index $s_i$ that reflects $i$’s relative cost burden in all possible jurisdictions and for all possible levels of public good production — and share indices are determined endogenously.\textsuperscript{4}

In this paper, we define the share equilibrium and provide an axiomatic characterization. Our characterization revolves around an extension of the consistency property of van den Nouweland, Tijs, and Wooders (2002). In the sense that the same axioms characterize the ratio equilibrium for public good economies and the share equilibrium, the concepts are closely related and the share equilibrium may be regarded as an extension of the ratio equilibrium

\textsuperscript{3}In some of the literature jurisdictions may be referred to as clubs.

\textsuperscript{4}In contrast, extensions of Samuelson’s concept of a Lindahl equilibrium to local public good economies depend on the use of personalized prices for each player for the local public good for each jurisdiction he might possibly join (cf. Wooders 1997) while extensions of the Walrasian equilibrium to local public goods economies depend on a price for each player for each jurisdiction that he may join and for each level of public good that can be produced in that jurisdiction (cf. Allouch and Wooders 2008). Cost-sharing notions of equilibrium to economies with local public goods as in, for example, Konishi, Le Breton and Weber (1997, 1998), treat equal cost sharing by members of each jurisdiction. These authors and many others, including the recent contribution of Gravel and Thoron (2007), also treat taxation schemes where the tax imposed on an individual is a percentage of his wealth.
to economies with local public goods. We show that a share equilibrium leads to jurisdictions and consumption levels of local public good and private good that are stable against group formation (that is, share equilibrium outcomes are in the core). To show that the share equilibrium is not an “empty” concept, we provide an existence of equilibrium result for symmetric local public good economies. To illustrate ideas and to relate our general model to commonly-encountered models in the local public good literature, we provide some examples; these demonstrate that our model and results allow the possibility of congestion and of multiple jurisdictions. To first allow the reader to gain some familiarity with the concept of a share equilibrium, we begin by relating the equilibrium to the core, we then treat special cases and examples, and conclude our analysis with the axiomatic characterization in Section 6.

2 Local public good economies

This section is devoted to formal definitions. We limit ourselves to economies with one public good and one private good.

In defining a local public goods economy we take as given a player set $N$ and a set of decision making players $D \subseteq N$. Until we introduce reduced economies the reader may think of $D$ as equal to $N$. The set of all possible jurisdictions, i.e., non-empty subsets of $N$, is denoted by $2^N \setminus \emptyset$ and the set of all jurisdictions containing a player $i \in N$ is denoted by $2^N(i)$.

The distinction between the sets $N$ and $D$ does not play any role in this and the next three sections of the paper (that is, for these sections we could simply let $D = N$). We could therefore write these sections with no mention of $D$. For our axiomatization in Section 6, however, we need to be able to distinguish between decision-making players and non-decision making players. We chose to introduce $D$ here rather than extending, in Section 6, definitions of concepts.
A local public good economy is a list

\[ E = (N, D; (w_i)_{i \in N}; (u_i)_{i \in N}; f_D), \]

where \( N \) (sometimes denoted \( N(E) \)) is the non-empty finite set of players in the economy, \( D \subseteq N \) is a set of decision makers, \( w_i \in \mathbb{R}_+ \) is the non-negative endowment of player \( i \in N \) of a private good, \( u_i : \mathbb{R}_+ \times \mathbb{R}_+ \times 2^{N(i)} \to \mathbb{R} \) is the utility function of player \( i \), and \( f_D : \mathbb{R}_+ \times 2^N \setminus \emptyset \to \mathbb{R}_+ \) is the cost function for the production of local public good in jurisdictions. The family of all public good economies is denoted by \( \mathcal{E} \).

In an economy \( E \), if player \( i \) is a member of jurisdiction \( J \subseteq N \) (\( i \in J \)) and consumes an amount \( x_i \) of the private good and an amount \( y \) of the local public good provided in jurisdiction \( J \), then player \( i \) enjoys utility \( u_i(x_i, y, J) \). We assume that \( u_i \) is strictly increasing in both private and (local) public good consumption.

In a jurisdiction \( J \), the cost of producing \( y \) units of the local public good borne by the decision-making players in \( J \) is \( f_D(y, J) \) units of the private good. We assume that the cost function \( f_D \) is non-decreasing in the level of (local) public good with \( f_D(0, J) = 0 \) for each jurisdiction \( J \).

A specification of a jurisdiction structure of the set of players, the levels of local public good provided in each jurisdiction, and private good consumptions is called a configuration. Hence, a configuration in a local public good economy \( E \) with set of players \( N \) is a vector

\[ (x, y, P) = ((x_i)_{i \in N}, (y_P)_{P \in \mathcal{P}}, \mathcal{P}), \]

where, for each \( i \in N \), \( x_i \in \mathbb{R}_+ \) is the consumption of the private good by player \( i \), \( \mathcal{P} \) is a partition of \( N \) into jurisdictions and, for each \( P \in \mathcal{P} \), \( y_P \in \mathbb{R}_+ \) is the level of local public good provided in jurisdiction \( P \subseteq N \).

We denote the set of configurations in a local public good economy \( E \) with
set of players $N$ by $C(N)$. A configuration $(x, y, P) \in C(N)$ is feasible if $f_D(y_P, P) \leq \sum_{i \in P \cap D} (w_i - x_i)$ for each $P \in P$.

3 Share equilibrium

A share equilibrium for a local public good economy consists of a vector of share indices - one for each decision-making player in the economy - and a configuration. Share indices determine the share of each decision-making player $i$ of the cost of the production of local public good in all jurisdictions containing player $i$; if a member of jurisdiction $J \subseteq N$, a decision-making player $i \in D$ with share index $s_i$ pays the share $s_i/\left(\sum_{j \in J \cap D} s_j\right)$ of the cost of local public good production in jurisdiction $J$. Hence, share indices determine the relative cost shares paid by the decision-making players in each jurisdiction that might possibly be formed. A set of share indices and a configuration constitute a share equilibrium if (1) every decision-making player’s membership of a jurisdiction and consumption as specified by the configuration are utility-maximizing in his budget set as determined by his (relative) share and, moreover, (2) in every jurisdiction that appears in an equilibrium, all decision-making members demand the same level of local public good. This implies that, in equilibrium, given the share of the cost of local public good production that he has to shoulder in various jurisdictions as determined by the share indices, each decision-making player prefers the jurisdiction to which he is assigned and the level of local public good that is provided in that jurisdiction. Moreover, in a share equilibrium, the players agree to share the cost of local public good production according to their share indices. Hence, a share equilibrium is an equilibrium in three dimensions: the cost shares arising from the players’ share indices, the jurisdictions formed, and a level of local public good production for each jurisdiction that
is formed. Agreement on the share indices determining cost shares, formation of jurisdictions, and levels of local public good are inextricably linked.

Formally, for a local public good economy $H = h; (z_l)_l = (x_l)_l$, and for each player $i \in D$ and each jurisdiction $J \subseteq N$, player $i$'s relative share in $J$ is $s_i^{J,D} = s_i/\left(\sum_{j \in J \cap D} s_j\right)$. Also, if $P$ is a partition of $N$, then for each $i \in N$ we denote the jurisdiction containing player $i$ by $P(i)$; thus $i \in P(i) \in P$.

A pair consisting of a set of share indices and a configuration $(s, (x, y, P))$ is a \textit{share equilibrium} in economy $E$ if for each $i \in D(E)$,\footnote{Assuming share indices to be positive is without loss of generality. It follows from condition 2 of share equilibrium and the assumption that players' utility functions are strictly increasing in local public good consumption that there can be no share equilibrium in which a player has a share index equal to zero as this would imply that a player always wants to consume more of the local public good, which is free to him. For similar reasons, it is impossible to have some players to be subsidized, i.e. have a negative share index.}

1. $s_i^{P(i),D} f_D(y_{P(i)}, P(i)) + x_i = w_i$, and,
2. for all $(\pi, \overline{y}, \overline{J}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times 2^N$ satisfying $i \in \overline{J}$ and $s_i^{\overline{J},D} f_D(\overline{y}, \overline{J}) + \pi_i \leq w_i$, it holds that $u_i(x_i, y_{P(i)}, P(i)) \geq u_i(\pi_i, \overline{y}, \overline{J})$.

The set of share equilibria of an economy $E = \langle N, D; (w_i)_{i \in N}; (u_i)_{i \in N}; f_D \rangle$ is denoted $SE(E)$.

\footnote{We could normalize the share indices to sum up to 1. However, this is unnecessary for any of our results and would complicate notation in the paper. Moreover, such a normalization is inherently arbitrary - for example, we may as well normalize the share indices such that they sum up to the number of players. Therefore, we have opted not to normalize the share indices at all, but let them be determined by the equilibrium conditions solely.}

\footnote{The equality in condition 1 could be “≤”, but using “=” is without loss of generality as an agent would always be able to consume more of the private good and therefore increase his utility if he wasn’t spending all of his initial endowment. Therefore condition 2 cannot be satisfied if $s_i^{P(i),D} f_D(y_{P(i)}, P(i)) + x_i < w_i$.}
Note that the share indices appear only in a relative manner so that, if \((s, (x, y, P))\) is a share equilibrium in an economy \(E\) and \(\alpha > 0\), then \((\alpha s, (x, y, P))\) is also a share equilibrium in economy \(E\).

4 Share equilibrium and the core

In this section we explore relations between share equilibria and the core of a local public good economy.

The core of an economy is the set of configurations that are stable against deviations by subsets of decision-making players, called coalitions. When a coalition deviates, its members can form new jurisdictions and within each of these jurisdictions the members can decide on a level of local public good to be provided and on a way to share the cost of its provision among the jurisdiction members. It is obvious that it will not be in the best interest of the members of a jurisdiction to subsidize the cost of the local public good in another jurisdiction, so that all jurisdictions will have to be self-sufficient. This implies that we can limit ourselves to considering deviations in which a deviating coalition forms one new jurisdiction that includes all members of the coalition, because a configuration is stable against such deviations if and only if it is stable against deviations by coalitions that can form multiple new jurisdictions amongst themselves.

Formally, the **core** of an economy \(E = \langle N, D; (w_i)_{i \in N}; (u_i)_{i \in N}; f_D \rangle\) is the set of all configurations \((x, y, P) \in C(N)\) such that for every \(T \subseteq N\), private-good consumption levels \((\overline{x}_i)_{i \in T}\), and local public good level \(\overline{y}\) satisfying \(f_D(\overline{y}, T) \leq \sum_{i \in T} (w_i - \overline{x}_i)\), it holds that, if there exists a player \(i \in T \cap D\) who is strictly better off (that is, a player \(i\) for whom \(u_i(x_i, y_{P(i)}, P(i)) < u_i(\overline{x}_i, \overline{y}, T)\)), then there exists a player \(j \in T \cap D\) who is strictly worse off (that is, a player \(i\) for whom \(u_j(x_j, y_{P(j)}, P(j)) > u_j(\overline{x}_j, \overline{y}, T)\)).
Theorem 1 shows that every share equilibrium leads to a configuration in the core. Thus, if a configuration is such that there exists a set of share indices that, together with that configuration, forms a share equilibrium, then that configuration is in the core of the economy. Results in the same spirit hold in other contexts.\(^9\) Since our model allows consumption and production possibilities to depend on the identities of individuals jointly producing and consuming the public good in each jurisdiction, which creates some new complexities, and since the share equilibrium is a new concept, we provide a proof.

**Theorem 1.** Let \(E = \langle N, D; (w_i)_{i \in N}; (u_i)_{i \in N}; f_D \rangle \in \mathcal{E}\) be a local public good economy, and let the share indices \((s_i)_{i \in D}\) and configuration \(( (x_i)_{i \in N}, (y_p)_{p \in P}, P)\) constitute a share equilibrium in economy \(E\). Then the configuration \((x, y, P)\) is in the core of \(E\).

**Proof.** Take a coalition \(T \subseteq N\) and an allocation of private and local public good consumption \(((\bar{x}_i)_{i \in T}, \bar{y})\) satisfying

\[
f_D(\bar{y}, T) \leq \sum_{i \in T \cap D} (w_i - \bar{x}_i).
\]

We distinguish between \(f_D(\bar{y}, T) = 0\) and \(f_D(\bar{y}, T) > 0\).

Suppose \(f_D(\bar{y}, T) = 0\) and suppose that for player \(i \in T \cap D\) it holds that \(u_i(x_i, y_{P(i)}, P(i)) < u_i(\bar{x}_i, \bar{y}, T)\). Then because \((s, (x, y, P)) \in SE(E)\), we know by condition 2 of share equilibrium that \(\bar{x}_i = s_i^{T:D} f_D(\bar{y}, T) + \bar{x}_i > w_i\).

\(^9\)To name a few: Debreu and Scarf (1963, Theorem 1) shows that price-taking equilibrium outcomes are in the core of a private-goods exchange economy; Foley (1970) shows that Lindahl equilibrium outcomes are in the core of an economy with public and private goods; Kaneko (1977b) notes that ratio equilibrium outcomes are in the core of a voting game in a public goods economy; Allouch and Wooders (2008) demonstrate such a result in the context of a club economy.
Together with $0 = f_D(\vec{y}, T) \leq \sum_{j \in T \cap D} (w_j - \pi_j)$, this implies that there exists a player $k \in T \cap D$ for whom $\pi_k < w_k$. However, player $k$ can afford to be in jurisdiction $T$ and consume $w_k$ of the private good and $\vec{y}$ of the local public good (because $s_k^{T,D} f_D(\vec{y}, T) + w_k = w_k$). Because $(s, (x, y, P))$ is a share equilibrium and the player’s utility function is strictly increasing in private good consumption, we can derive

$$u_k(x_k, y_{P(k)}, P(k)) \geq u_k(w_k, y, T) > u_k(\pi_k, y, T).$$

Hence, $k$ would be strictly worse off by belonging to $T$ with the proposed allocation. We conclude that $(x, y, P)$ is stable against deviations $((\pi_i)_{i \in T}, \vec{y}, T)$ with $f_D(\vec{y}, T) = 0$.

Now suppose $f_D(\vec{y}, T) > 0$. Then $\sum_{i \in T \cap D} (w_i - \pi_i) > 0$, and we can induce relative share indices for $T$ given by

$$\tau_i := \frac{w_i - \pi_i}{\sum_{j \in T \cap D} (w_j - \pi_j)}$$

for all $i \in T \cap D$. Note that $\sum_{i \in T \cap D} \tau_i = 1$ and $\tau_i f_D(\vec{y}, T) + \pi_i \leq w_i$ for all $i \in T \cap D$.

We compare the induced relative share indices for the decision-making members of $T$, $(\tau_i)_{i \in T \cap D}$, to their relative indices according to $s$, which are given by

$$s_i^{T,D} := \frac{s_i}{\sum_{j \in T \cap D} s_j}$$

for each $i \in T \cap D$. We distinguish two cases.

**Case 1.** Suppose that $s_i^{T,D} = \tau_i$ for all $i \in T \cap D$. Then, for all $i \in T \cap D$,

$$s_i^{T,D} f_D(\vec{y}, T) + \pi_i = \tau_i f_D(\vec{y}, T) + \pi_i \leq w_i.$$

By condition 2 of the share equilibrium, we then know that $u_i(x_i, y_{P(i)}, P(i)) \geq u_i(\pi_i, \vec{y}, T)$ for all $i \in T \cap D$. Hence, no decision-making player in $T$ is strictly
better off in jurisdiction $T$ with consumption bundle $(\overline{x}_i, \overline{y})$ than he is in configuration $(x, y, P)$.

**Case 2.** Suppose that it does not hold that $s^T_i = \tau_i$ for all $i \in T \cap D$. Since $\sum_{j \in T \cap D} s^T_j = \sum_{j \in T \cap D} \tau_j = 1$, it holds that $s^{T,D}_k < \tau_k$ for some $k \in T \cap D$. Such a player $k$ would loose from belonging to $T$ with the proposed allocation. To see this, first observe that

$$w_k \geq \tau_k f_D(\overline{y}, T) + \overline{x}_k > s^{T,D}_k f_D(\overline{y}, T) + \overline{x}_k.$$  

Consider

$$\overline{x}_k := w_k - s^{T,D}_k f_D(\overline{y}, T) > \overline{x}_k.$$  

Then

$$s^{T,D}_k f_D(\overline{y}, T) + \overline{x}_k = w_k.$$  

It follows from condition 2 of the share equilibrium $(s, (x, y, P))$ and the assumption that a player’s utility function is strictly increasing in private good consumption that

$$u_k(x_k, y_{P(k)}, P(k)) \geq u_k(\overline{x}_k, \overline{y}, T) > u_k(\overline{x}_k, \overline{y}, T).$$  

Hence, $k$ would be strictly worse off by belonging to $T$ with the proposed allocation.

Based on Case 1 and Case 2, we conclude that $(x, y, P)$ is also stable against deviations $((\overline{x}_i)_{i \in T}, \overline{y}, T)$ with $f_D(\overline{y}, T) > 0$. ■

### 5 Existence of a share equilibrium

Now that we have identified this new equilibrium concept, there are, of course, many questions that arise. One question is under what conditions
existence of an equilibrium can be insured. Here we provide an existence theorem for symmetric economies and leave a more in-depth study of existence of share equilibrium for future work.

A local public good economy is symmetric if all its players play the same role and have the same preferences and endowments. Note that, because players who share the same jurisdiction enter into one another’s utility functions, symmetry necessarily implies that players care only about how many players are in their jurisdiction and not about the identities of those members. It also implies that costs for producing the local public good in a jurisdiction only depend on how many players are in the jurisdiction. Precisely, a local public good economy $E = \langle N; D; (w_i)_{i \in N}; (u_i)_{i \in N}; f_D \rangle \in \mathcal{E}$ is symmetric if all players participate as decision makers ($D = N$) and there exist:

(a) $w \in \mathbb{R}_+$ such that $w_i = w$ for all $i \in N$,

(b) a function $f : \mathbb{R}_+ \times \mathbb{N} \to \mathbb{R}_+$ such that $f_D(y, J) = f(y, |J|)$ for all $y \in \mathbb{R}_+$ and $J \subseteq N$,\(^{10}\) and

(c) a function $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{N} \to \mathbb{R}$ such that for each player $i \in N$, each jurisdiction $J \subseteq N$ with $i \in J$, each amount of private good $x_i$ and amount of local public good $y_i$ it holds that $u_i(x_i, y_i, J) = u(x_i, y_i, |J|)$.

We will denote a symmetric local public good economy by $E = \langle N; w; u; f \rangle$, with notations as above.

Let $E = \langle N; w; u; f \rangle$ be a symmetric local public good economy in which the utility and cost functions are both continuous.\(^{11}\) In addition, assume that

\(^{10}\) $\mathbb{N}$ denotes the set of positive integers and, for any finite set $J$, $|J|$ denotes the number of elements of $J$.

\(^{11}\) Note that, because jurisdiction sizes are integers, the condition of continuity has no bite for the third argument in the utility function or the second argument in the cost function.
the cost function is such that no jurisdiction can afford an unlimited level of local public good, i.e., for each jurisdiction size \( k \) there exists a level of local public good \( y \) such that \( f(y, k) > kw \). We denote the set of symmetric local public good economies in \( E \) that satisfy these continuity and cost conditions by \( \mathcal{ES} \).

Let \( E = \langle N; w; u; f \rangle \in \mathcal{ES} \). Also, let \( k \in \mathbb{N} \). There exists a largest level of local public good \( y(k) \) satisfying \( f(y(k), k) = kw \); this follows from the assumptions that (i) the cost function \( f \) is continuous, (ii) \( f(0, k) = 0 \), (iii) \( f \) is non-decreasing in \( y \), and (iv) there exists a level of local public good \( y \) such that \( f(y, k) > kw \). We define a level of local public good

\[
y_k \in \arg \max_{y \in [0, y(k)]} ku \left( w - \frac{1}{k} f(y, k), y, k \right).
\]

Since the set \([0, y(k)]\) is compact and since \( u \) is continuous, such a \( y_k \) exists. The quantity \( y_k \) is a level of local public good that maximizes the total utility of the players in a jurisdiction of size \( k \) when the players in the jurisdiction share the costs of local public good production equally. We denote this maximum total utility for a jurisdiction of size \( k \) by \( U(k) \);

\[
U(k) = ku \left( w - \frac{1}{k} f(y_k, k), y_k, k \right).
\]

Now, for each coalition of players \( S \subseteq N \), we define \( v^E(S) \) to be the maximum total utility obtainable by the players in a coalition \( S \) when they consider forming various jurisdictions not including any players not in \( S \) and when the costs of local public good production are shared equally in each possible jurisdiction;

\[
v^E(S) = \max_{P \in \mathcal{P}(S)} \sum_{J \in P} U(|J|),
\]

where \( \mathcal{P}(S) \) denotes the set of all partitions of \( S \). Note that a coalition of players may have to split up into multiple jurisdictions to obtain the maximal utility \( v^E(S) \) for its members.
We say that the economy $E$ is top convex if the function $v^E$ satisfies the condition that $\frac{v^E(S)}{|S|} \leq \frac{v^E(N)}{|N|}$ for all $S \subseteq N$. Top convexity means that, among all configurations in the economy that are symmetric within each of their jurisdictions, the per-capita utility of the players in the economy is maximal in some configuration that maximizes the sum of all players’ utilities.\footnote{A related condition that is also called top convexity was defined in a setting of networks in Jackson and van den Nouweland (2005), who in turn got their inspiration from a condition called ”domination by the grand coalition” that was defined for coalitional games by Chatterjee et al. (1993). This condition, which stipulates that the per-capita value is maximal for the grand coalition, was identified but unnamed by Shubik (1982, p.149) as a necessary and sufficient requirement for the non-emptiness of the core of a coalitional game with transferable utility. Our condition differs in that, as in much of the local public good literature, we allow for congestion and multiple jurisdictions and consider the maximum per capita utility over all partitions of the set of players into jurisdictions.} Note that the condition of top convexity allows congestion; it is possible that $v^E(N)$ is achieved by some non-trivial partition of players into jurisdictions. Informally, top convexity implies that the player set $N$ can be partitioned into ‘optimal’ jurisdictions. We illustrate the top convexity condition in the following example. The example is chosen in part to demonstrate how models in the literature on economies with local public goods, such as Allouch and Wooders (2008) and its antecedents, fit into the framework used in the current paper.

**Example 1** A top convex, symmetric local public good economy. Assume that all players are identical in terms of their endowments, preferences, and effects on others. Each player’s utility is additively separable in private and local public good consumption on one hand and jurisdiction membership on the other hand. The utility for private and local public good consumption takes a Cobb-Douglas form and the utility for jurisdiction membership depends on jurisdiction size, so that it has the interpretation of a congestion or crowding
function. The cost of local public good production is taken such that the *per capita* cost are constant across levels of local public good and jurisdictions. This is a special case that often appears in the literature and that makes the good a *local public service*. We may think of the local public good as, for example, some service offered by a condominium for which the costs are fixed in per capita terms.\footnote{For example, the local public good could be window-cleaning in the condominium. The condominium association of one of the authors provides this service from condominium association fees.}

Formally, let $E = \langle N; w; u; f \rangle$ be a symmetric local public good economy with $u(x, y, J) = x^\alpha y^{1-\alpha} - v(|J|)$, where $\alpha \in (0, 1)$, $v: \mathbb{N} \to \mathbb{R}$ is the anonymous congestion or crowding function, and $f(y, J) = |J| y$.\footnote{Note that with these definitions the economy satisfies the continuity and cost conditions of \( E \mathcal{S} \).}

We choose the crowding function $v$ so that only jurisdictions of size 2 or 3 are desirable:

$$v(k) = \begin{cases} 
0 & \text{if } k = 2 \text{ or } k = 3 \\
1 & \text{otherwise} 
\end{cases}$$

We start by computing $y_k$ for a jurisdiction consisting of $k$ players.

$$ku \left( w - \frac{1}{k} f(y, k), y, k \right) = k \left( (w - y)^\alpha y^{1-\alpha} - v(k) \right),$$

where $v(k)$ is either 0 or 1. The level of local public good that maximizes this expression is equal to $y_k = (1 - \alpha)w$, which implies that in this example the optimal level of local public good is independent of the size of the jurisdiction.\footnote{Of course, this is a feature that follows from the way in which we set up this particular example to make it easily tractable. It is not true in general for a symmetric local public good economy.}

We can now compute the maximum total utility $U(k)$ of the players in a jurisdiction of size $k$. We simplify notation by defining $A = \alpha^\alpha (1 - \alpha)^{1-\alpha} w,$
a quantity that is independent of the size of the jurisdiction.

\[
U(k) = k (w - (1 - \alpha)w^\alpha((1 - \alpha)w)^{1 - \alpha} - v(k)) = kA - kv(k).
\]

Note that the first component in this expression results in a per-capita utility \(A\) that is constant across jurisdictions of various sizes, whereas the second component varies with jurisdiction size on a per-capita basis.

Because of the way in which we defined the crowding function \(v\), it follows that a coalition of players reaches the maximum total utility by breaking itself up into jurisdictions of size 2 or 3. This can be done for any coalition size except for a coalition consisting of a single player: A coalition with an even number \(s\) of players can be broken up into \(\frac{s}{2}\) jurisdictions of size 2 and a coalition with an odd number \(s \geq 3\) of players can be broken up into 1 jurisdiction of size 3 and \(\frac{s-3}{2}\) jurisdictions of size 2. Using this, we obtain

\[
v^E(S) = \begin{cases} 
A - 1 & \text{if } |S| = 1 \\
|S|A & \text{if } |S| > 1 
\end{cases}
\]

Thus,

\[
\frac{v^E(S)}{|S|} = \begin{cases} 
A - 1 & \text{if } |S| = 1 \\
A & \text{if } |S| > 1 
\end{cases}
\]

and the economy \(E\) is top convex because the maximum per-capita utility can be obtained for the coalition consisting of all players (by splitting itself up into two or three person jurisdictions). □

Theorem 2 shows that a share equilibrium exists for every top convex symmetric local public good economy with continuous utility and cost functions in which no jurisdiction can afford an unlimited level of local public good.
Theorem 2. Let $E = \langle N; w; u; f \rangle \in \mathcal{ES}$ be a symmetric local public good economy that is top convex. Then $SE(E) \neq \emptyset$.

Proof. Define share indices $s_i$ by $s_i = 1$ for each $i \in N$, so that all players have the same share index. Also, let

$$P(N) \in \arg \max_{P \in \mathcal{P}(N)} \sum_{J \in P} (u(|J|)).$$

For each jurisdiction $J \in P(N)$, choose a

$$y_J \in \arg \max_{y \in [0,y(|J|)]} |J|u \left( w - \frac{1}{|J|}f(y,|J|), y, |J| \right).$$

For each $i \in N$, denote the jurisdiction in $P(N)$ containing player $i$ by $J(i)$ and define a level of private good consumption by

$$x_i = w - \frac{1}{|J(i)|}f(y_{J(i)},|J(i)|).$$

We will prove that the share indices $s$ and the configuration $((x_i)_{i \in N}, (y_J)_{J \in P(N)}, P(N))$ form a share equilibrium in the symmetric economy $E$.

Claim 1. $\sum_{J \in P(N)} v^E(J) = v^E(N)$. This follows from the following string of (in)equalities.

$$\sum_{J \in P(N)} v^E(J) = \sum_{J \in P(N)} \left( \max_{P \in \mathcal{P}(J)} \sum_{K \in P} \left( |K|u \left( w - \frac{1}{|K|}f\left(y_{|K|}, |K| \right), y_{|K|}, |K| \right) \right) \right) \leq \max_{P \in \mathcal{P}(N)} \sum_{K \in P} \left( |K|u \left( w - \frac{1}{|K|}f\left(y_{|K|}, |K| \right), y_{|K|}, |K| \right) \right) = v^E(N) = \sum_{J \in P(N)} \left( |J|u \left( w - \frac{1}{|J|}f\left(y_{|J|}, |J| \right), y_{|J|}, |J| \right) \right) \leq \sum_{J \in P(N)} \left( \max_{P \in \mathcal{P}(J)} \sum_{K \in P} \left( |K|u \left( w - \frac{1}{|K|}f\left(y_{|K|}, |K| \right), y_{|K|}, |K| \right) \right) \right) = \sum_{J \in P(N)} v^E(J).$$
Claim 2. \( v^E(J) = |J|u \left( w - \frac{1}{|J|}f \left( y_{|J|}, |J| \right) \right) \) for all \( J \in P(N) \).

This follows from the fact that the last (weak) inequality in the sequence of (in)equalities above is an equality, as we have just derived.

Claim 3. \( \frac{v^E(J)}{|J|} = \frac{v^E(N)}{|N|} \) for all \( J \in P(N) \). To see this, we derive

\[
v^E(N) = \sum_{i \in N} \frac{v^E(N)}{|N|} = \sum_{J \in P(N)} \sum_{i \in J} \frac{v^E(N)}{|N|} = \sum_{J \in P(N)} |J| \frac{v^E(N)}{|N|} \geq \sum_{J \in P(N)} v^E(J),
\]

where the last inequality follows because top convexity of \( E \) implies that \( |J| \frac{v^E(N)}{|N|} \geq v^E(J) \) for each \( J \in P(N) \). Because \( \sum_{J \in P(N)} v^E(J) = v^E(N) \) by Claim 1, it follows that all the weak inequalities are in fact equalities, so that we can derive that \( |J| \frac{v^E(N)}{|N|} = v^E(J) \) for each \( J \in P(N) \).

We are now ready to prove that \((s, ((x_i)_{i \in N}, (y_J)_{J \in P(N)}, P(N)))\) is a share equilibrium. First note that for each potential jurisdiction the costs of local public good production are shared equally among all jurisdiction members because all players participate in the decision-making process and have the same share index. Condition 1 of the share equilibrium then follows from \( x_i = w - \frac{1}{|J(i)|}f \left( y_{J(i)}, |J(i)| \right) \). To show that condition 2 of the share equilibrium also holds, fix \( i \in N \) and let \((\bar{x}_i, \bar{y}, \bar{J}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times 2^N \) be such that \( i \in \bar{J} \) and \( \frac{1}{|J|}f(\bar{y}, |J|) + \bar{x}_i \leq w \). We need to show that \( u_i(x_i, y_{J(i)}, J(i)) \geq u_i(\bar{x}_i, \bar{y}, \bar{J}) \).

First, note that \( f(\bar{y}, |J|) \leq |J| (w - \bar{x}_i) \leq |J| w \), from which it follows that \( \bar{y} \leq y(\bar{J}) \). Using this, we derive

\[
u_i(\bar{x}_i, \bar{y}, \bar{J}) \leq u(w - \frac{1}{|J|}f(\bar{y}, |J|), \bar{y}, |J|) \leq \frac{1}{|J|} |J| u(w - \frac{1}{|J|}f(\bar{y}, |J|), \bar{y}, |J|) \leq \frac{1}{|J|} v^E(\bar{J}) \leq \frac{v^E(N)}{|N|} \]
\[ v^E(J(i)) = \frac{\sum_{J(i)} f(J(i))}{|J(i)|} = \alpha \cdot w = \frac{w - 1}{|J(i)|} f(J(i)) = u(x_i, y_{J(i)}, J(i)) \]

where the first inequality follows from utility being strictly increasing in private good consumption, the second inequality follows from the definition of \( y_H(M_l) \), the third inequality follows from the definition of \( v^E(J) \), the fourth inequality follows from top convexity of \( E \), the first equality follows by Claim 3, the second equality follows from Claim 2, the third equality follows from the definition of \( y_{J(i)} \), and the last equality follows from the definition of \( x_i \).

We illustrate Theorem 2 in the following example.

**Example 2** A share equilibrium for a symmetric economy. Consider the top convex symmetric local public good economy in example 1. We remind the reader that we computed \( y_k = (1 - \alpha)w \) for each \( k \in \mathbb{N} \) and

\[ U(k) = kA - kv(k), \]

where \( A = \alpha^\alpha(1 - \alpha)^{1-\alpha}w \). We also determined that a coalition of players reaches the maximum total utility by breaking itself up into jurisdictions of size 2 or 3. Because the one-player economy is uninteresting, assume that \( E \) has at least two players.

The share equilibria of this economy identified in Theorem 2 consist of a share index \( s_i = 1 \) for each player \( i \) and a configuration \((x, y, P)\), where \( P \) is a partition of \( N(E) \) into jurisdictions of size 2 or 3, \( y_J = (1 - \alpha)w \) for each \( J \in P \), and

\[ x_i = w - \frac{1}{|J(i)|} f(y_{J(i)}, J(i)) = \alpha w \]
for each player $i$. Note that we are identifying multiple share equilibrium configurations. For example, if there are 6 players, then we have identified share equilibria with each of the 10 possible partitions of $N(E)$ into 3 jurisdictions of size 2 and also with each of the 10 possible partitions of $N(E)$ into 2 jurisdictions of size 3. ■

Of course, an economy does not have to be symmetric for it to have a share equilibrium. In an economy with players of more than one type (heterogeneous players), the properties of a share equilibrium will depend on the specifics of the utility functions of the various players and the costs of producing local public good in various jurisdictions. For example, players may explicitly value diversity and, keeping consumption levels fixed, derive a higher utility from being in a jurisdiction with different types of players than from being among players of the same type. If this is the case, a share equilibrium may exist in which jurisdictions contain multiple types of players. On the other hand, if the cost function for public good production is such that it is relatively more expensive to produce local public good for a mixed jurisdiction than for a homogeneous jurisdiction, then we may find a share equilibrium in which the players segregate themselves into homogeneous jurisdictions. As stated before, we postpone a comprehensive study of existence of share equilibrium to future work. Below, we provide two examples that simply illustrate the possible variety in share equilibria.

**Example 3** Existence of an equilibrium with heterogenous jurisdictions. We consider an economy very similar to that in Example 1, but assume that there are two types of players, who differ only in one respect: each player prefers the company of the opposite type to that of their own type.\(^{16}\)

\(^{16}\)The interpretation of the two types as “men” and “women” comes to mind.
Each player (of either type) has an endowment \( w \) and utility function

\[
\begin{align*}
    u(x_i, y, J) &= x_i^\alpha y^{1-\alpha} - v(J),
\end{align*}
\]

where \( v \) is the congestion or crowding function, and \( f(y, J) = |J| y \). We choose the crowding function \( v \) so that only mixed jurisdictions of size 2 are desirable

\[
v(J) = \begin{cases} 
    0 & \text{if } J \text{ contains one player of type 1 and one player of type 2} \\
    1 & \text{otherwise}
\end{cases}
\]

We give each player a share index \( s_i = 1 \). Then we find, similar to Example 1, that for a fixed jurisdiction, each player achieves the maximum utility possible by consuming \( y^* = (1 - \alpha)w \) of the local public good and \( x_i^* = \alpha w \) of the private good. Hence, defining \( A = \alpha^\alpha (1 - \alpha)^{1-\alpha}w \), the maximum utility a player can obtain as a member of a jurisdiction \( J \) equals

\[
u_i^* (x_i^*, y^*, J) = A - v(J).
\]

Note that the first component in this expression is constant across jurisdictions, so that the second component becomes decisive for the choice of jurisdiction.

Hence, we find that a share equilibrium exists when there are equal numbers of players of each of the two types, so that a partition of the players into mixed jurisdictions of size 2 can be accomplished. Note that this conclusion depends on the assumption that players have equal share indices. \( \blacksquare \)

Because we do not want to leave the reader with the impression that all share equilibria have to involve the same share index for each player, we end this section with another example.

**Example 4** A share equilibrium in which players have different share indices.

We consider an economy very similar to that in Example 1, but assume that there are two players, who differ only in their relative utility from private and local public good consumption.

Both players 1 and 2 have an endowment \( w \) and congestion and production costs are anonymous: \( v(J) = 2 - |J| \) and \( f(y, J) = |J| y \) for each
jurisdiction $J$. Player 1 has utility function $u_1(x_1, y, J) = x_1^\alpha y^{1-\alpha} - v(J)$, and player 2 has utility function $u_2(x_2, y, J) = x_2^\beta y^{1-\beta} - v(J)$.

Players’ optimal consumption levels in singleton jurisdictions are independent of their share indices (because they bear the total cost of public good production if they are the only jurisdiction member). In jurisdiction $J^1 = \{1\}$, player 1’s utility-maximizing consumption levels are $y^1 = (1 - \alpha)w$ of the public good and $x^1 = \alpha w$ of the private good, so that player 1 can get a maximum utility of $u_1(x^1, y^1, J^1) = \alpha^\alpha(1 - \alpha)^{(1-\alpha)w-1}$. Player 2’s utility-maximizing consumption levels in jurisdiction $J^2 = \{2\}$ are $y^2 = (1 - \beta)w$ and $x^2 = \beta w$, which result in utility $u_2(x^2, y^2, J^2) = \beta^\beta(1 - \beta)^{(1-\beta)w-1}$.

Now, suppose the players’ share indices are $v_1$ and $v_2$ and consider the possible jurisdiction $J^3 = \{1, 2\}$. In this jurisdiction, player 1 pays the share $s_1^3 = \frac{v_1}{s_1 + s_2}$ of the cost of local public good production and player 2 pays the rest. We simplify notation and denote player 1’s share in jurisdiction $J^3$ by $s_1^3 = \frac{v_1}{s_1 + s_2}$, leaving player 2’s share as $s_2^3 = 1 - s_1^3$.

Player 1’s utility-maximizing levels of consumption in jurisdiction $J^3$ are found as the solution to

$$\begin{align*}
\text{maximize} & \quad x_1^\alpha y^{1-\alpha} \\
\text{subject to} & \quad x_1 + 2s_1^3y = w
\end{align*}$$

The solution to this problem is $x_1^3 = \alpha w$ and $y_1^3 = \frac{(1-\alpha)w}{2s_1^3}$. Similarly, we find that player 2’s optimal consumption levels in jurisdiction $J^3$ are $x_2^3 = \beta w$ and $y_2^3 = \frac{(1-\beta)w}{2s_2^3}$.

To find a share equilibrium in which jurisdiction $J^3$ is formed, we need the demand for local public good in the jurisdiction to be the same for both players. Thus, $\frac{(1-\alpha)w}{2s_1^3} = \frac{(1-\beta)w}{2s_2^3}$ needs to hold. Using $s_2^3 = 1 - s_1^3$, we find that this requirement leads to $s_1^3 = \frac{1-\alpha}{2-\alpha-\beta}$ and, consequently, $s_2^3 = \frac{1-\beta}{2-\alpha-\beta}$. So, for example, we can choose the share indices $s_1 = 1 - \alpha$ and $s_2 = 1 - \beta$.

\[\text{Note that } v(J^3) = 0 \text{ and } f(y, J^3) = 2y.\]
With these share indices, both players demand $y^3 = \frac{(2-\alpha-\beta)w}{2}$ of the public good and they have utilities $u_1(x_1^3, y^3, J^3) = (\alpha)^a \left( \frac{(2-\alpha-\beta)}{2} \right)^{1-\alpha} w$ and $u_2(x_2^3, y^3, J^3) = (\beta)^b \left( \frac{(2-\alpha-\beta)}{2} \right)^{1-\beta} w$.

Thus, we find a share equilibrium with share indices $s_1 = 1 - \alpha$ and $s_2 = 1 - \beta$ in which jurisdiction $\{1, 2\}$ is formed and $\frac{(2-\alpha-\beta)w}{2}$ public good is produced as long as for each player the utility in this jurisdiction is higher than that obtainable by forming a singleton jurisdiction. The conditions for this to hold are $(\alpha)^a \left( \frac{(2-\alpha-\beta)}{2} \right)^{1-\alpha} w \geq \alpha^a(1 - \alpha)^{(1-\alpha)}w - 1$ and $(\beta)^b \left( \frac{(2-\alpha-\beta)}{2} \right)^{1-\beta} w \geq \beta^b(1 - \beta)^{(1-\beta)}w - 1$. It depends on the parameters of the economy if these conditions are satisfied, but it is clear that it is not impossible that they are. For example, the conditions are satisfied if $\alpha = \frac{1}{4}$, $\beta = \frac{3}{4}$, and $w = 5$. ■

6 An axiomatization of the share equilibrium

It is apparent that $SE$ is a special case of a mapping $\phi$ that assigns to each public good economy $E \in \mathcal{E}$ a set of pairs each consisting of a vector of numbers and a configuration, that is,

$$\phi(E) \subseteq \mathbb{R}^{D(E)} \times C(N(E)).$$

We will call such a mapping a solution on $\mathcal{E}$. We consider various properties of such solutions and show that these properties axiomatically characterize the share equilibrium. At the heart of our axiomatizations is the notion of consistency.
6.1 Consistency

Consistency dictates that, when the same solution concept is used in all groups, solutions reached in subgroups of players should be aligned with those reached in the group consisting of all players. To express this idea formally we must first consider how to reduce an economy to a subgroup $R$ of players. The introduction of a set of decision makers in the definition of an economy enables us to create a reduced economy (itself satisfying our definition of a local public goods economy) whose solution can be studied.

We define a reduced economy with player set $R$ taking as given the share indices of members of $N \setminus R$. Suppose that the decision-making players in a local public good economy agree on their share indices and a configuration. The agreement is consistent if, taking the share indices of the players in $N \setminus R$ as given, no subgroup $R$ of players has an incentive to change their part of the agreement – their own share indices, their jurisdictions of membership, or levels of local public good for those jurisdictions. Notice that, in interpretation, the players in $N \setminus R$ do not leave the economy but only the decision-making process. In the reduced economy, share indices of the members of $N \setminus R$, which determine their cost shares of local public good production in jurisdictions, are taken as given.

We now formally introduce reduced economies. Take as given a local public good economy $E = \langle N, D; (w_i)_{i \in N}; (u_i)_{i \in N}; f_D \rangle$. Let $R \subseteq D$ be a subset of decision makers, $R \neq \emptyset$, and let $(s, (x, y, P)) \in \mathbb{R}^{D(E)} \times C(N(E))$. The reduced economy of $E$ with respect to $R$ and $(s, (x, y, P))$ is the economy in which the set of decision-making players is $R$, so

$$E^R,(s,(x,y,P)) = \langle N, R; (w_i)_{i \in N}; (u_i)_{i \in N}, fr \rangle,$$

Think of $R$ as standing for "reduced-economy decision-making agents".

\[18\]
and where the production function $f_R$ satisfies

$$f_R(\bar{y}, J) = \left[ \sum_{i \in J \cap R} s_i^{J,D} \right] f_D(\bar{y}, J)$$

for all $\bar{y} \in \mathbb{R}_+$ and $J \subseteq N$.¹⁹

The idea behind the definition of the reduced economy is as follows. Suppose all decision-making players agree on the share indices $s$ and configuration $(x, y, P)$. This implies that they agree on cost-sharing schemes corresponding to the share indices $s$, formation of local jurisdictions $P$, and levels of local public good production for each of those jurisdictions. Then, if, in addition to the players in $N \setminus D$, the players in $D \setminus R$ also withdraw from the decision-making process, the players in $R$ can reconsider the jurisdictions that they form, the levels of local public good to be produced in those jurisdictions, and their relative shares of the residual cost of producing the local public good. When they reconsider, they take into account that the players in $G \setminus U$ have agreed to their share indices and will pay their corresponding shares for the cost of local public good production. Hence, when reconsidering the cost-sharing scheme, the players in $R$ face the residual cost

$$f_R(\bar{y}, J) = \left[ 1 - \sum_{i \in J \cap (D \setminus R)} s_i^{J,D} \right] f_D(\bar{y}, J) = \left[ \sum_{i \in J \cap R} s_i^{J,D} \right] f_D(\bar{y}, J)$$

for producing a level $\bar{y}$ of local public good in a jurisdiction $J$.

¹⁹It is interesting to note that an issue to be resolved in defining consistency of a solution concept is to ask what elements are player-specific and remain fixed for the members of $N \setminus R$. In their study of the Walrasian equilibrium for a private goods economy, van den Nouweland et al. (1996) keep fixed the allocations of those players leaving the economy and axiomatize Walrasian allocations. In the current paper we are axiomatizing equilibrium cost shares and thus, analogously, the cost shares of those individuals withdrawing from the decision-making process are fixed.
A solution is consistent if, once agreement on relative cost shares has been reached, the withdrawal of some players from the decision-making process will not influence the final outcome of the process. The consistency property is defined using reduced economies. A solution $\phi$ on $\mathcal{E}$ is consistent (CONS) if it satisfies the following condition.

If $E \in \mathcal{E}$, $(s, (x, y, P)) \in \phi(E)$, and $R \subset D(E)$, $R \neq \emptyset$, then $E^{R,(s,(x,y,P))} \in \mathcal{E}$ and $(s^R, (x, y, P)) \in \phi(E^{R,(s,(x,y,P))})$.

Here, $s^R = (s_i)_{i \in R}$ denotes the vector of share indices of the members of $R$.

We illustrate reduced economies and the role of the share indices in consistency with the following example.

**Example 5.** Decision making players cannot change cost shares of non decision-making players. Suppose that we have an economy with 3 players in which all players are decision-makers; $N = D = \{1, 2, 3\}$. Suppose the players agree on share indices $s_1 = s_2 = 1$ and $s_3 = 2$. Then, if jurisdiction $J_1 = \{1, 2\}$ is formed and a level of local public good $y$ is produced in this jurisdiction, player 1’s cost share would be $s_{1}^{J_1,D} = \frac{1}{2}$, so that player 1 would pay $\frac{1}{2}f_D(y, J_1)$. In jurisdiction $J_2 = \{1, 3\}$, player 1’s cost share would be $s_{1}^{J_2,D} = \frac{1}{3}$ and he would pay $\frac{1}{3}f_D(y, J_2)$, and in jurisdiction $J_3 = N$, player 1 would pay $\frac{1}{4}f_D(y, J_3)$.

Now, suppose that player 1 leaves the decision-making process, agreeing to his share index $s_1 = 1$. In the reduced economy, the set of decision-making players is $R = \{2, 3\}$. The cost function $f_R(\cdot, \cdot)$ for the reduced economy is different from the cost function $f_D(\cdot, \cdot)$ only for jurisdictions that include player 1. Specifically, $f_R(y, J_1) = \frac{1}{2}f_D(y, J_1)$, $f_R(y, J_2) = \frac{2}{3}f_D(y, J_2)$, and $f_R(y, J_3) = \frac{3}{4}f_D(y, J_3)$. The players in $R$ can now reconsider not only the jurisdictions that they want to form and the levels of local public good for
those jurisdictions, but also their share indices. Suppose that player 3’s share index is changed to \( \tilde{\nu}_3 = 1 \). This changes player 3’s relative standing vis-à-vis player 2 and shifts costs from player 3 to player 2. For example, in jurisdiction \( J_4 = \{2, 3\} \), player 3 will now have to pay the cost \( \frac{1}{2} f_R(y, J_4) = \frac{1}{2} f_D(y, J_4) \), whereas before he changed his share index, he would have had to pay \( \frac{2}{3} f_R(y, J_4) = \frac{2}{3} f_D(y, J_4) \). However, note that changing his share index does not allow player 3 to put a larger share of the (cost) burden of producing local public good on the non-decision-making player 1, as player 1’s share of the cost in various jurisdictions was agreed upon before he left the decision-making process. For example, if player 3 now wants to form jurisdiction \( J_2 \) with player 1, then player 3 will still have to shoulder the cost \( f_R(y, J_2) = \frac{2}{3} f_D(y, J_2) \). In jurisdiction \( J_3 \) with all three players, players 2 and 3 pay \( \frac{1}{2} f_R(y, J_3) = \frac{2}{5} f_D(y, J_3) \) each. \( \blacksquare \)

The share equilibrium is a consistent solution, as is shown in the following lemma.

**Lemma 1.** The share equilibrium on the family \( \mathcal{E} \) of local public good economies is consistent.

**Proof.** Let \( E = \langle N, D; (w_i)_{i \in N}; (u_i)_{i \in N}; f_D \rangle \in \mathcal{E} \) be a local public good economy, let \( (s, (x, y, P)) \in SE(E) \), and let \( R \subseteq D \), \( R \neq \emptyset \). Let \( f_R \) be the cost function of the reduced economy \( E^{R,(s,(x,y,P))} \); that is,

\[
f_R(\overline{y}, J) = \left[ \sum_{j \in J \cap R} s_i^{J,D} \right] f_D(\overline{y}, J) \geq 0
\]

for every \( \overline{y} \in \mathbb{R}_+ \) and \( J \subseteq N \). Obviously, \( E^{R,(s,(x,y,P))} \in \mathcal{E} \).
For all $i \in R$ and $J \subseteq N$ with $i \in J$ it holds that
\[
s^i_{D}f_D(\overline{y}, J) = \frac{s_i}{\sum_{j \in J \cap R} s_j} f_D(\overline{y}, J) = \frac{s_i}{\sum_{j \in J \cap R} s_j} \left[ \sum_{j \in J \cap R} s^j_D \right] f_D(\overline{y}, J) = \frac{s_i}{\sum_{j \in J \cap R} s_j} f_R(\overline{y}, J) = s^i_{R}f_R(\overline{y}, J)
\]
for all $\overline{y} \in \mathbb{R}_+$. We now derive
\[
s^i_{P(i),R}f_R(y_{P(i)}, P(i)) + x_i = s^i_{P(i),D}f_D(y_{P(i)}, P(i)) + x_i = w_i.
\]

Let $(\overline{x}, \overline{y}, \overline{J}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times 2^N$ be such that $i \in \overline{J}$ and $s^i_R f_R(\overline{y}, \overline{J}) + \overline{x_i} \leq w_i$. Then $s^i_{T,D}f_D(\overline{y}, \overline{J}) + \overline{x_i} \leq w_i$ and, because $(s, (x, y, P)) \in SE(E)$, we know that $u_i(x_i, y_{P(i)}, P(i)) \geq u_i(\overline{x}, \overline{y}, \overline{J})$. This proves that $(s^R, (x, y, P)) \in SE(E^{R,(s,(x,y,P))})$. ■

### 6.2 An axiomatization using consistency

The share equilibrium for local public good economies can be axiomatized using the consistency property introduced in the previous subsection and two additional axioms – converse consistency and one-person rationality.

Whereas consistency states that agreements that are acceptable for the group of all players should be acceptable in all smaller groups as well, converse consistency states that if a set of share indices and a configuration constitute an acceptable solution for all proper subgroups of players, then they also constitute an acceptable solution for the group as a whole. Formally, a solution $\phi$ on $E$ is **converse consistent** (COCONS) if, for every $E \in \mathcal{E}$ with at least two players ($|N(E)| \geq 2$) and for every set of share indices $s = (s_i)_{i \in D(E)} \in \mathbb{R}^{D(E)}$ and every configuration $(x, y, P) = ((x_i)_{i \in N}, (y_P)_{P \in \mathcal{P}}, \mathcal{P})$, 
the following condition is satisfied.

If \( E \in \mathcal{E} \) and for every \( R \subset D(E) \) with \( R \not\in \{\emptyset, D(E)\} \) it holds that \( E^{R,(s,(x,y,P))} \in \mathcal{E} \) and \( (s^R, (x,y,P)) \in \phi(E^{R,(s,(x,y,P))}) \),

then \( (s, (x,y,P)) \in \phi(E) \).

This means that for an economy with 3 players who are all decision-makers, for example, if a set of share indices and a configuration induce a share equilibrium in all 1- and 2-player reduced economies, then they must form a share equilibrium in the 3-player economy. Hence, we can determine if a vector of share indices and a configuration form a share equilibrium by checking the reduced economies.

It is shown in the following lemma that the share equilibrium satisfies converse consistency.

**Lemma 2.** The share equilibrium on the family \( \mathcal{E} \) of local public good economies satisfies converse consistency.

**Proof.** Let \( E = (N, D; (w_i)_{i \in N}; (u_i)_{i \in N}; f_D) \in \mathcal{E} \) be a local public good economy with \(|N| \geq 2\) and let share indices \( s = (s_i)_{i \in D} \in \mathbb{R}^D \) and configuration \( (x, y, P) = ((x_i)_{i \in N}, (y_P)_{P \in P}, P) \in C(N) \) be such that, for every \( R \subset D \) with \( R \not\in \{\emptyset, D\} \), it holds that \( E^{R,(s,(x,y,P))} \in \mathcal{E} \) and \( (s^R, (x,y,P)) \in SE(E^{R,(s,(x,y,P))}) \). Then \( (s^{(i)}, (x,y,P)) \in SE(E^{(i),(s,(x,y,P))}) \) for each \( i \in D \).

Let \( i \in D \) and let \( f_{(i)} \) be the cost function of the reduced economy \( E^{(i),(s,(x,y,P))} \); that is, \( f_{(i)}(\overline{y}, J) = s_i^{LD} f_D(\overline{y}, J) \) for all \( \overline{y} \in \mathbb{R}_+ \) and \( J \subseteq N \) with \( i \in J \). Note that \( s_i^{(i)} = 1 \) for all \( J \subseteq N \) with \( i \in J \). Applying conditions 1 and 2 of share equilibrium to \( (s^{(i)}, (x,y,P)) \), we find that \( f_{(i)}(y_P(i), P(i)) + x_i = w_i \) and \( u_i(x_i, y_P(i), P(i)) \geq u_i(\overline{x}_i, \overline{y}, \overline{J}) \) for all \( (\overline{x}_i, \overline{y}, \overline{J}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times 2^N \) satisfying \( i \in \overline{J} \) and \( f_{(i)}(\overline{y}, \overline{J}) + \overline{x}_i = w_i \). Using \( f_{(i)}(\overline{y}, J) = s_i^{LD} f_D(\overline{y}, J) \) for all \( \overline{y} \in \mathbb{R}_+ \) and \( J \subseteq N \) with \( i \in J \), we then see that conditions 1 and 2 of share equilibrium are satisfied when applied to \( (s, (x,y,P)) \) and player \( i \).
Since we chose player \( i \in D \) arbitrarily, we can conclude that conditions 1 and 2 of share equilibrium are satisfied for \((s, (x, y, P))\) and each \( i \in D \), which proves that \((s, (x, y, P)) \in SE(E)\). 

Consistency and converse consistency link solutions in larger economies to those for smaller economies and vice versa. One-person rationality considers solutions with one decision-making player only. A solution \( \phi \) on \( E \) satisfies one-person rationality (OPR) if, for every local public good economy with one decision-making player \( E = \langle N, \{i\}; (w_j)_{j \in N}; (u_j)_{j \in N}; f_{\{i\}} \rangle \in E \), it holds that

\[
\phi(E) = \{(s_i, (x, y, P)) \mid s_i > 0, f_{\{i\}}(y_{P(i)}, P(i)) + x_i = w_i, \\
\text{and } u_i(x_i, y_{P(i)}, P(i)) \geq u_i(\pi_i, \eta, \tau) \text{ for all } (\pi_i, \eta, \tau) \\
satisfying } i \in \tau \text{ and } f_{\{i\}}(\eta, \tau) + \pi_i \leq w_i\}.
\]

The one-person rationality axiom dictates that in a one-decision-maker economy, the single decision-making player maximizes his utility given his endowment of the private good and the cost of producing certain amounts of local public good in various jurisdictions when the player has to pay all the residual cost of local public good provision. This is a rationality assumption much like those that prevail throughout economics.

The interaction of the three axioms consistency, converse consistency, and one-person rationality is explained in the following lemma.

**Lemma 3.** Let \( \phi \) and \( \psi \) be two solutions on \( E \) that both satisfy one-person rationality. If \( \phi \) is consistent and \( \psi \) is converse consistent, then it holds that \( \phi(E) \subseteq \psi(E) \) for all \( E \in E \).

*Proof.* We will prove the lemma by induction on the number of players. If \( E \in E \) is an economy with one decision-making player - \(|D(E)| = 1\) - then it follows from OPR of \( \phi \) and \( \psi \) that \( \phi(E) = \psi(E) \).
Now, let $E \in \mathcal{E}$ be an economy with $n$ decision-making players and suppose that it has already been proven that $\phi(E) \subseteq \psi(E)$ for all economies with fewer than $n$ decision-making players. Let $(s, (x, y, P)) \in \phi(E)$. Then, by CONS of $\phi$, we know that $E^{R,(s,(x,y,P))} \in \mathcal{E}$ and $(s^R, (x, y, P)) \in \phi(E^{R,(s,(x,y,P))})$ for all $R \subseteq D(E)$, $R \notin \{\emptyset, D(E)\}$. Hence, it follows from the induction hypothesis that $(s^R, (x, y, P)) \in \psi(E^{R,(s,(x,y,P))})$ for all $R \subseteq D(E)$, $R \notin \{\emptyset, D(E)\}$. So, by COCONS of $\psi$, we know that $(s, (x, y, P)) \in \psi(E)$. We conclude that $\phi(E) \subseteq \psi(E)$. 

Theorem 3 shows that consistency, converse consistency, and one-person rationality characterize the share equilibrium.

**Theorem 3.** The share equilibrium is the unique solution on $\mathcal{E}$ that satisfies one-person rationality, consistency, and converse consistency.

**Proof.** In Lemmas 1 and 2 we proved that the share equilibrium satisfies CONS and COCONS. To show that the share equilibrium satisfies OPR, let $E = \langle N, \{i\}; (w_j)_{j \in N}; (u_j)_{j \in N}; f_{\{i\}} \rangle \in \mathcal{E}$ be a local public good economy with one decision-making player. Note that in an economy with one decision-making player, the single decision-making player present will have to pay fully the remaining cost for each level of local public good that he wants to have available in a jurisdiction. The players’ share index plays no role in this case as there are no other decision-making players so that it needs not be determined how to share the cost of local public good production among decision-making players. Hence, the decision-making player simply determines for each possible jurisdiction the optimal level of local public good within his budget and chooses the jurisdiction and local public good combination that maximizes his utility. This is exactly what OPR describes.

To prove uniqueness, assume that $\phi$ is a solution on $\mathcal{E}$ that also satisfies the three foregoing axioms. Let $E \in \mathcal{E}$ be arbitrary. Then, Lemma 3 shows
that \( \phi(E) \subseteq SE(E) \) by CONS of \( \phi \) and COCONS of the share equilibrium, and that \( SE(E) \subseteq \phi(E) \) by CONS of the share equilibrium and COCONS of \( \phi \). Hence, \( \phi(E) = SE(E) \). ■

We conclude this section with the remark that the three axioms used to characterize the share equilibrium in Theorem 3 are logically independent. This is easily seen by considering the following three solutions on \( \mathcal{E} \). First, consider a solution \( \phi \) on \( \mathcal{E} \) that gives each decision-making player the same share index, assigns players to jurisdictions arbitrarily, has a level of local public good equal to 0 for each jurisdiction formed, and let each player consume his entire initial endowment (so that no player pays for local public good provision). Define \( \phi \) by \( \phi(E) = \{(s, (x, y, P)) \mid s_i = 1 \text{ and } x_i = w_i \text{ for each } i \in N(E), \text{ } P \text{ is a partition of } N(E), \text{ and } y_P = 0 \text{ for all } P \in \mathcal{P} \} \). This solution satisfies CONS and COCONS, but fails to satisfy OPR. Second, consider the solution \( \chi \) on \( \mathcal{E} \) that coincides with the share equilibrium for economies with one decision-making player, and is empty for economies with more than one decision-making player. Formally, \( \chi \) is defined by \( \chi(E) = SE(E) \) if \( |D(E)| = 1 \) and \( \chi(E) = \emptyset \) if \( |D(E)| > 1 \). This solution satisfies OPR and CONS, but does not satisfy COCONS. Finally, consider the solution \( \psi \) on \( \mathcal{E} \) that coincides with the share equilibrium for economies with one decision-making player, and for economies with more than one decision-making player assigns arbitrary share indices to players, groups them into jurisdictions arbitrarily, and has levels of local public good and of private-good consumption that are such that in each jurisdiction formed its members pay for the public good provided. So, \( \psi \) is defined by \( \psi(E) = SE(E) \) if \( |D(E)| = 1 \) and \( \psi(E) = \{(s, (x, y, P)) \mid s \in \mathbb{R}^{D(E)}_+, \text{ } P \text{ is a partition of } N(E), \text{ } x_i \leq w_i \text{ for all } i \in N(E) \text{ and } \sum_{i \in P \cap D}(w_i - x_i) = f_D(y_P, P) \text{ for each } P \in \mathcal{P} \} \) if \( |D(E)| > 1 \). This solution satisfies OPR and COCONS, but does not satisfy CONS.
7 Discussion

There are many situations in which people are obliged to pay their share of the cost of a public good according to some sharing formula, even when they do not participate in the decision-making process. One example is given by condominium home owners association agreements in which owners of units within the condominium contract to pay shares, sometimes based on relative sizes of units owned, of costs of goods and services paid for by the association.\footnote{One of the authors recently considered purchasing a townhouse in a complex where the commons and exterior of the units are maintained by a home owners association. The covenants governing the association, by which each owner of a unit in the complex is legally bound, stipulate that during any meeting of the members at which at least 50\% of all the votes in the association are represented, either is person or by proxy, a majority vote of those represented is binding on all members. Among other things, the members can decide to take on new projects and to increase members’ contributions to the home owners association to cover the projects’ costs. The covenants also stipulate that any such contributions have to be paid by each member within 30 days and (financial) penalties are spelled out for those not paying within 30 days. The covenants governing the home owners association to which the other author belongs are similar.} If the method of reaching agreements is consistent, condominium owners need not be concerned if they are unable to attend a meeting of the home owners association, because when reaching agreements in meetings, attendees will have no incentive to overturn agreements that may have been reached in informal discussions.\footnote{Then why do association members go to home owners meetings? This question was posed at a presentation of this research at the University of Warwick. Legal requirements (in the states of Oregon and Tennessee, at least) stipulate that a quorum must be present at the meeting to make binding decisions on the association members. We conjecture that it is this stipulation that motivates meetings in the evenings with refreshments.}

To summarize, in this paper, we introduced the concept of a share equilibrium for local public good economies where relative share indices of players
determine their shares of costs of local public good provision in each possible jurisdiction to which they may belong. The relative cost shares, and hence the relative share indices of players who end up in the same jurisdiction, can be determined just by considering the differences between their endowment and their consumption of private good. Note, however, that the share indices also contain information on relative cost shares of players in hypothetical jurisdictions, that is, ones that do not appear in the equilibrium jurisdiction structure. In pure public good economies, where all players are by definition in the same jurisdiction, there is no need to consider hypothetical jurisdictions and the cost shares of players in such jurisdictions. Hence, for pure public good economies, we can suffice by specifying players’ actual cost shares, or their relative share in the unique jurisdiction. In this manner, we obtain the ratio equilibrium for pure public good economies, as defined in Kaneko (1977a,b) and studied in van den Nouweland, Tijs, and Wooders (2002), as a special case of the share equilibrium. If all players have the same share index the equal cost sharing of Konishi, Le Breton and Weber (1998), for example, is another special case.

Since a share equilibrium is a new concept, there are many open questions. In the current paper, we show that a share equilibrium leads to jurisdictions and consumption levels of local public good and private good that are in the core of an economy and we prove existence of share equilibria for a class of symmetric local public good economies. We also provide an axiomatic characterization of the share equilibrium. This gives us insight into the properties of this equilibrium concept and shows that it is in the same spirit as the ratio equilibrium. In ongoing research we continue to address questions on existence of share equilibrium and the relationship between share equilibrium and Lindahl equilibrium, as well as other questions relating to share equilibrium.
References


