On the rank of payoff matrices with long-term assets.

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Abstract

We consider a stochastic financial exchange economy with a finite date-event tree representing time and uncertainty and a nominal financial structure with possibly long-term assets. We exhibit a sufficient condition under which the return matrix and the full return matrix have the same rank. This generalizes previous results of Angeloni-Cornet and Magill-Quinzii involving only short-term assets. We then derive existence results with assumptions only based on the fundamentals of the economy.

Keywords: Incomplete Markets, financial equilibrium, multi-period model, long-term assets.

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Contents

1 Introduction 3

2 The $T$-period financial exchange economy 4
   2.1 Time and uncertainty 4
   2.2 The financial structure 5

3 On the ranks of the return matrices 6

4 Existence of equilibrium 15
   4.1 The stochastic exchange economy 15
   4.2 Financial equilibrium 16
   4.3 No-arbitrage and financial equilibrium 16
   4.4 Existence of equilibrium 17

5 Appendix 22
1 Introduction

In this paper, we consider a standard model of stochastic financial exchange economy with a finite date-event tree representing time and uncertainty as described in [3]. The financial structure is composed of a finite set of nominal assets with possibly long-term assets.

The financial structure is formally represented by the payoff matrix, which provides for each asset and each node, the return of this asset in this node. For a simpler presentation, we adopt the convention of Angeloni and Cornet [1] where an asset is issued at one date and never retraded after. We refer to this article to show how to transform a financial structure with possible retrading as considered in [3] into an equivalent financial structure without retrading.

When we consider the asset markets at different dates and nodes and the asset prices on these markets, we need to define a full payoff matrix to determine the budget constraints of an agent. The full payoff matrix is derived from the payoff matrix by adding on the column corresponding to an asset and on the row corresponding to the issuance node of this asset, the opposite of the price of the asset on the financial market.

Now, with the full payoff matrix, we can define no-arbitrage prices and characterize them by the existence of positive state prices such that the asset prices are the present value of the payoffs.

In a two period economy, it is well known and actually quite obvious that the rank of the payoff and full payoff matrices are equal when the asset prices do not exhibit arbitrage. This means that the ranges of the possible wealth transfers with the payoff and the full payoff matrices have the same dimension. In particular, we can characterize a complete financial structure only by knowing the rank of the return matrix. Indeed, if the rank is equal to the number of states in the second period, whatever are the no-arbitrage asset prices, the full payoff matrix is also of maximal rank and then all transfers compatible with the state prices are feasible through the financial structure.

With more than two periods, the above result is no more true as shown in [3, 1]. Below we provide a simple numerical example where the payoff matrix has a maximal rank and the full payoff matrix does not for some no arbitrage asset prices. So, for some asset prices, the financial structure is complete and for some others it is not. It is then no more possible to characterize a complete financial structure only with the payoff matrix.

Furthermore, the dependance of the rank of the full return matrix from no arbitrage asset prices may lead to the failure of the existence of a financial equilibrium. See an example in [3]. In a two period economy, this kind of phenomenon appears with real assets due to the drop of the rank on the return matrix but never with nominal assets.

Actually, in [3] and [1], it is shown that the ranks of the payoff and full payoff matrices are equal when the assets are all short term. A short term asset is characterized by the fact that it has non zero return only at the immediate successors of its issuance node. If an asset is not short term, it is called a long term asset.
Our main purpose is this paper is to tackle the question of the ranks of the return matrices with long term assets and to obtain existence result of financial equilibria under assumptions more tractable than those of [1].

After introducing notations and the model of a financial structure in Section 2, we provide a sufficient condition, Assumption R, in Section 3, to get the equality of the ranks of the payoff and full payoff matrices for non arbitrage asset prices. We show that Assumption R is satisfied if all assets are short term, if there is a unique issuance date, or if there is no overlap of the nodes with non zero returns for two different assets. More generally, Assumption R translates the fact that the assets issued at a given node are true financial innovations in the sense that the payoffs cannot be replicated by assets issued before.

In Section 4, we consider a stochastic financial exchange economy with possibly long term nominal assets and we provide several existence results when Assumption R is satisfied by the payoff matrix. These results are based on the existence result (Theorem 3.1) of [1] but Assumption R allows us to replace an abstract assumption in [1] by a more verifiable assumption on the return matrix. Furthermore, our result then holds true for any state prices.

In [1], it is mentioned that the abstract condition is satisfied with only short term assets. So our contribution could be seen as the extension to long term assets under Assumption R.

The differences between our different results come from the fact that we can consider more general portfolio sets translating restricted participations when the return matrix is of maximal rank whereas when we have redundant assets, we get only existence when the agents have a full access to the asset markets.

2 The T-period financial exchange economy

In this section, we present the model and the notations, which are borrowed from Angeloni-Cornet[1] and are essentially the same as those of Magill-Quinzii[3].

2.1 Time and uncertainty

We\(^1\) consider a multi-period exchange economy with \((T + 1)\) dates, \(t \in \mathcal{T} := \{0, \ldots, T\}\), and a finite set of agents \(\mathcal{I}\). The uncertainty is described by a date-event tree \(\mathcal{D}\) of length \(T + 1\). The set \(\mathcal{D}_t\) is the set of nodes (also called date-

\(^1\)We use the following notations. A \((\mathcal{D} \times \mathcal{J})\)-matrix \(A\) is an element of \(\mathbb{R}^{\mathcal{D} \times \mathcal{J}}\), with entries \((a(\xi, j))_{(\xi, j) \in \mathcal{D} \times \mathcal{J}}\); we denote by \(A_{\xi} \in \mathbb{R}^{\mathcal{J}}\) the \(\xi\)-th row of \(A\) and by \(A^j \in \mathbb{R}^{\mathcal{D}}\) the \(j\)-th column of \(A\). We recall that the transpose of \(A\) is the unique \((\mathcal{J} \times \mathcal{D})\)-matrix \(A^t\) satisfying \((A x) \bullet y = x \bullet (A^t y)\) for every \(x \in \mathbb{R}^{\mathcal{J}}, y \in \mathbb{R}^{\mathcal{D}}\), where \(\bullet\) denotes the usual inner product in \(\mathbb{R}^{\mathcal{J}}\) (resp. \(\mathbb{R}^{\mathcal{D}}\)). We denote by \(\text{rank}A\) the rank of the matrix \(A\) and by \(\text{Vect}(A)\) the range of \(A\), that is the linear sub-space spanned by the column vectors of \(A\). For every subset \(\mathcal{D} \subset \mathcal{D}\) and \(\mathcal{J} \subset \mathcal{J}\), the matrix \(A^{\mathcal{D} \times \mathcal{J}}\) is the \((\mathcal{D} \times \mathcal{J})\)-sub-matrix of \(A\) with entries \(a(\xi, j)\) for every \((\xi, j) \in (\mathcal{D} \times \mathcal{J})\). Let \(x, y \in \mathbb{R}^n; x \geq y\) (resp. \(x \gg y\)) means \(x_h \geq y_h\) (resp. \(x_h > y_h\)) for every \(h = 1, \ldots, n\) and we let \(\mathbb{R}^n_{\geq} = \{x \in \mathbb{R}^n : x \geq 0\}\), \(\mathbb{R}^n_{\gg} = \{x \in \mathbb{R}^n : x \gg 0\}\). We also use the notation \(x \gg y\) if \(x \geq y\) and \(x \neq y\). The Euclidean norm in the Euclidean different spaces is denoted \(\|\cdot\|\) and the closed ball centered at \(x\) and of radius \(r > 0\) is denoted \(B(x, r) := \{y \in \mathbb{R}^n \mid \|y - x\| \leq r\}\).
events) that could occur at date \( t \) and the family \((D_t)_{t \in T}\) defines a partition of the set \( D \); for each \( \xi \in D \), we denote by \( t(\xi) \) the unique date \( t \in T \) such that \( \xi \in D_t \).

At date \( t = 0 \), there is a unique node \( \xi_0 \), that is \( D_0 = \{ \xi_0 \} \). As \( D \) is a tree, each node \( \xi \in D \setminus \{ \xi_0 \} \) has a unique immediate predecessor denoted \( pr(\xi) \) or \( \xi^- \). The mapping \( pr \) maps \( D_t \) to \( D_{t-1} \). Each node \( \xi \in D \setminus D_T \) has a set of immediate successors defined by \( \xi^+ = \{ \xi \in D : \xi = \xi^- \} \).

For \( \tau \in T \setminus \{ 0 \} \) and \( \xi \in D \setminus \bigcup_{t=0}^{\tau-1} D_t \), we define \( pr^\tau(\xi) \) by the recursive formula: \( pr^\tau(\xi) = pr(p r^{\tau-1}(\xi)) \). We then define the set of successors and the set of predecessors of \( \xi \) as follows:

\[
D^+ \xi \; \{ \xi' \in D : \exists \tau \in T \setminus \{ 0 \} | \xi = pr^\tau(\xi') \}
D^- \xi \; \{ \xi' \in D : \exists \tau \in T \setminus \{ 0 \} | \xi' = pr^\tau(\xi) \}
\]

If \( \xi' \in D^+ (\xi) \) [resp. \( \xi' \in D^+ (\xi) \cup \{ \xi \} \)], we shall use the notation \( \xi' > \xi \) [resp. \( \xi' \geq \xi \)]. Note that \( \xi' \in D^+ (\xi) \) if and only if \( \xi \in D^- (\xi') \) and similarly \( \xi' \in \xi^+ \) if and only if \( \xi = (\xi')^- \).

### 2.2 The financial structure

The financial structure is constituted by a finite set of assets denoted \( J = \{ 1, \ldots, J \} \). An asset \( j \in J \) is a contract issued at a given and unique node in \( D \) denoted \( \xi(j) \), called emission node of \( j \). Each asset is bought or sold only at its emission node \( \xi(h) \) and yields payoffs only at the successor nodes \( \xi' \) of \( D^+ (\xi(j)) \).

To simplify the notation, we consider the payoff of asset \( j \) at every node \( \xi \in D \) and we assume that it is zero if \( \xi \) is not a successor of the emission node \( \xi(j) \). The payoff may depend upon the spot price vector \( p \in \mathbb{R}^L \) and is denoted by \( V^j (p, \xi) \). Formally, we assume that \( V^j (p, \xi) = 0 \) if \( \xi \notin D^+ (\xi(j)) \).

For each consumer, \( z_i = (z_i^j)_{j \in J} \in \mathbb{R}^J \) is called the portfolio of agent \( i \). If \( z_i^j > 0 \) [resp. \( z_i^j < 0 \)], then \( |z_i^j| \) is the quantity of asset \( j \) bought [resp. sold] by agent \( i \) at the emission node \( \xi(j) \).

We assume that each consumer \( i \) is endowed with a portfolio set \( Z_i \subset \mathbb{R}^J \), which represents the set of admissible portfolios for agent \( i \). For a discussion on this concept we refer to Angeloni-Cornet [1], Aouani-Cornet [4] and the references therein.

To summarize a financial structure \( F = (J, (Z_i)_{i \in I}, (\xi(j)), j \in J, V) \) consists of

- a set of assets \( J \),
- a collection of portfolio sets \((Z_i \subset \mathbb{R}^J)_{i \in I}\),
- a node of issuance \( \xi(j) \) for each asset \( j \in J \),
- a payoff mapping \( V : \mathbb{R}^L \rightarrow \mathbb{R}^{D \times J} \) which associates to every spot price \( p \in \mathbb{R}^L \) the \((D \times J)\)-payoff matrix \( V(p) = (V^j (p, \xi))_{\xi \in D, j \in J} \) and satisfies the condition \( V^j (p, \xi) = 0 \) if \( \xi \notin D^+ (\xi(j)) \).
The price of asset $j$ is denoted by $q_j$; it is paid at its emission node $\xi(j)$. We let $q = (q_j)_{j \in J} \in \mathbb{R}^J$ be the asset price vector.

The full payoff matrix $W(p, q)$ is the $(D \times J)$-matrix with the following entries:

$$W^j(p, q)(\xi) := V^j(p, \xi) - \delta_{\xi, \xi'} q_j,$$

where $\delta_{\xi, \xi'} = 1$ if $\xi = \xi'$ and $\delta_{\xi, \xi'} = 0$ otherwise.

So, given the prices $(p, q)$, the full flow of returns for a given portfolio $z \in \mathbb{R}^J$ is

$$W(p, q)z = \sum_{j \in J} V^j(p, \xi_j) z_j - \sum_{j \in J} \delta_{\xi, \xi'} q_j z_j,$$

where $\delta_{\xi, \xi'} = 1$ if $\xi = \xi'$ and $\delta_{\xi, \xi'} = 0$ otherwise.

We now recall that for a given spot price $p$, the asset price $q$ is a strongly no-arbitrage price if it does not exist a portfolio $z \in \mathbb{R}^J$ such that $W(p, q)z > 0$. The characterization of strongly no-arbitrage price (see, e.g. Magill-Quinzii [3]) states that $q$ is a strongly no-arbitrage price if and only if it exists a state price vector $\lambda \in \mathbb{R}_+^D$ such that $W(p, q)\lambda = 0$. Taken into account the particular structure of the matrix $W(p, q)$, this is equivalent to

$$\forall j \in J, \lambda_{\xi(j)} q_j = \sum_{\xi \in D} \lambda_{\xi} V^j(p, \xi).$$

### 3 On the ranks of the return matrices

In two period model, the rank of $V(p)$ and of $W(p, q)$ are equals when the price $q$ is a no-arbitrage price. Indeed, the matrix $W(p, q)$ is simply built from $V(p)$ by replacing the first row by the transpose of $-q$. Then, since the no-arbitrage condition implies that $q$ is a positive linear combination of the rows of $V(p)$, we easily conclude.

As already noticed in Magill and Quinzii [3] and in Angeloni and Cornet [1], this result is no more true if the number of dates is strictly larger than 2. Let us give an example with a maximal rank return matrix. Let a financial structure with three dates, each non-terminal node has two immediate successors so that $D = \{\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8\}$. At each non-terminal node, two assets are issued, hence $J = 6$. The return matrix $V$ is constant and equal to

$$V = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}.$$
One remarks that the rank of the matrix $V$ is 6. We now consider the asset price $q = (7, 7, 2, 1, 1, 1)$. Hence the full-return matrix is

$$W(q) = \begin{bmatrix}
-7 & -7 & 0 & 0 & 0 & 0 \\
1 & 2 & -2 & -1 & 0 & 0 \\
1 & 1 & 0 & 0 & -1 & -1 \\
2 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}$$

One easily check that $q$ is a no-arbitrage price since $^tW(q)\lambda = 0$ with $\lambda = (1, 1, 1, 1, 1, 1, 1) \in \mathbb{R}^7_+$. But the rank of $W(q)$ is 5 since the dimension of the kernel of $W(q)$ is 1. So, even if the rank of $V$ is maximal, then for the no-arbitrage price $q$, the financial market is incomplete in the sense that the rank of the full return matrix is strictly lower than $\#\mathcal{D} - 1$. This phenomenon cannot appear in a two-date model. Then the rank of $V$ is no more sufficient to determine if the market is complete or not and the completeness may depend on the asset price with more than two periods.

Our next example exhibits a converse paradox. Indeed the return matrix $V$ is not of maximal rank but the market is complete for a well chosen no-arbitrage price $q$. The date-event tree is the same as in the above example and the number of assets as well as the dates of issuance are also identical. But now the return matrix is

$$V = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}$$

One easily check that the rank of $V$ is 5 strictly smaller than the maximal rank which is 6. Let $q = \left(\frac{5}{2}, \frac{5}{2}, 1, 2, 2, 1\right)$. The full return matrix is then:

$$W(q) = \begin{bmatrix}
-\frac{5}{2} & -\frac{5}{2} & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & -2 & 0 & 0 \\
1 & 1 & 0 & 0 & -2 & -1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}$$

The price $q$ is a no-arbitrage price since $^tW(q)\lambda = 0$ with

$$\lambda = \left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}\right) \in \mathbb{R}^7_+.$$
But the rank of $W(q)$ is 6, that is the maximal rank and then the market is complete.

The discrepancy of the ranks between the return matrix and the full return matrix is not only a problem to determine what is a complete market for a multi-period economy but, as shown in an example of Magill-Quinzii [3], it also leads to the non-existence of equilibrium even with a nominal asset structure where the return matrix $V$ does not depend on the spot price $p$.

In Magill-Quinzii [3] and in Angeloni-Cornet [1], it is shown that the discrepancy of the ranks cannot happen when there are only short-lived assets, that is the payoff of an asset $j$ issued at node $\xi$ is zero except at the immediate successor of $\xi$.

The main purpose of this section is to provide a sufficient condition compatible with long term assets under which the rank of the return matrix is equal to the rank of the full return matrix for every no-arbitrage asset price.

We first introduce some additional notations. For all $\xi \in \mathbb{D} \setminus \mathbb{D}_T$, $\mathcal{J}(\xi)$ is the set of assets issued at the node $\xi$, that is $\mathcal{J}(\xi) = \{ j \in \mathcal{J} \mid \xi(j) = \xi \}$ and $\mathcal{J}(\mathbb{D}^- (\xi))$ is the set of assets issued at a predecessor of $\xi$, that is $\mathcal{J}(\mathbb{D}^- (\xi)) = \{ j \in \mathcal{J} \mid \xi(j) < \xi \}$. For all $t \in \{1, \ldots, T+1\}$, we denote by $\mathcal{J}_t$ the set of assets issued at date $t$, that is, $\mathcal{J}_t = \{ j \in \mathcal{J} \mid \xi(j) \in \mathbb{D}_t \}$.

Let $(\tau_1, \ldots, \tau_k)$ such that $0 \leq \tau_1 < \tau_2 < \ldots < \tau_k \leq T - 1$ be the dates at which there is at least the issuance of one asset, that is $\mathcal{J}_{\tau_k} \neq \emptyset$. For $\kappa = 1, \ldots, k$, let $\mathbb{D}_{\tau_{\kappa}}$ be the set of nodes at date $\tau_{\kappa}$ at which there is the issuance of at least one asset. $\mathbb{D}_{\tau}$ is the set of nodes at which there is the issuance of at least one asset. We remark that

$$\bigcup_{\tau \in \{0, \ldots, T-1\}} \mathcal{J}_\tau = \bigcup_{\kappa \in \{1, \ldots, k\}} \mathcal{J}_{\tau_{\kappa}} = \mathcal{J}, \quad J = \sum_{\kappa \in \{1, \ldots, k\}} \# \mathcal{J}_{\tau_{\kappa}}$$

and for all $\tau \in \{\tau_1, \ldots, \tau_k\}$, $\bigcup_{\xi \in \mathbb{D}_\tau} \mathcal{J}(\xi) = \mathcal{J}_\tau$.

In the remainder of this section, we consider a fixed spot price $p$ so that we remove it as an argument of $V$ for the sake of simpler notations.

We now state our assumption on the matrix of returns and its consequence on the rank of the matrix $V$ and $W(q)$.

**Assumption R.** $\forall \kappa \in \{2, \ldots, k\}, \forall \xi \in \mathbb{D}_{\tau_{\kappa}},$

$$\text{Vect} \left( V^\mathcal{J}(\mathbb{D}^- (\xi)) \right) \cap \text{Vect} \left( V^\mathcal{J}(\xi) \right) = \{0\}.$$

This assumption means that the returns of the assets issued at a node $\xi$ are not redundant with the returns of the assets issued at a predecessor node of $\xi$. So, the issuance of additional assets at $\xi$ are a true financial innovation since the payoffs in the successors of $\xi$ cannot be replicated by the payoffs of a portfolio built with the assets issued before $\xi$.

In the following lemma, we show that if the assumptions R holds true for the financial structure $\mathcal{F}$, it is also true for any financial substructure $\mathcal{F}'$ of $\mathcal{F}$ obtained by considering only a subset $\mathcal{J}'$ of the set of asset $\mathcal{J}$.
Lemma 1. Let $\mathcal{F} = (\mathcal{J}, (Z_i)_{i \in I}(\xi(j)))_{j \in \mathcal{J}}, V)$ and $\mathcal{F}' = (\mathcal{J'}, (Z'_i)_{i \in I}(\xi(j)))_{j \in \mathcal{J'}}, V')$ two financial structures such as $\mathcal{J}' \subset \mathcal{J}$. If Assumption $R$ holds true for the structure $\mathcal{F}$ then it holds also true for the structure $\mathcal{F}'$.

Proof. The proof of Lemma 1 is given in Appendix.

Remark 1. The converse of Lemma 1 is not true. Let us consider an economy with three periods such as: $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}\}$. There are three assets issued at nodes 0, 1 and 2. The return matrix is:

$$V = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}$$

One remark that Assumption R is not satisfied for $V$ whereas it holds true for the reduced financial structure where we keep only the two first assets and for which the return matrix is

$$V = \begin{bmatrix}
0 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}$$

The next proposition provides some sufficient conditions under which Assumption R holds true.

Proposition 1. The return matrix $V$ satisfies Assumption R if one of the following condition is satisfied:

(i) For all $j \in \mathcal{J}$, asset $j$ is a short term asset in the sense that $V^j(\xi') = 0$ if $\xi' \notin \mathcal{J}^+$.

(ii) All assets are issued at the same date $\tau_1$.

(iii) For all $\xi \in \mathbb{D}^e$, $\mathcal{D}^+(\xi) \cap \mathbb{D}^e = \emptyset$, which means that if an asset is issued at node $\xi$, then no assets is issued in a successor of $\xi$.

(iv) For all $(\xi, \xi') \in (\mathbb{D}^e)^2$, if $\xi < \xi'$, then $V_{\mathcal{J}(\xi')} = 0$, which means that if an asset $j$ is issued at node $\xi$ and another one at a successor $\xi'$, then the return of $j$ at the successors of $\xi'$ are equal to 0.

The proof of this proposition is left to the reader. It is a consequence of the fact that either $\mathcal{J}(\mathbb{D}^-(\xi))$ is an empty set or $\text{Vect} \left( V_{\mathcal{J}(\mathbb{D}^+(\xi))} \right) = \{0\}$.

Now, we state the main result of this section:
Proposition 2. If the return matrix $V$ satisfies Assumption R, then for all no-arbitrage price $q$, $\text{rank} V = \text{rank} W(q)$.

Condition (i) of Proposition 1 shows that Proposition 2 is a generalization of Proposition 5.2. b) et c) in Angeloni-Cornet [1] and of Magill-Quinzii [3] where only short-term assets are considered.

Remark 2. For the following financial structure, Assumption R does not hold true and yet, for any (no-arbitrage or not) price of assets $q$, $\text{rank} V = \text{rank} W(q)$. So Assumption R is sufficient but not necessary. Let us consider the date-event tree $D = \{\xi_0, \xi_1, \xi_2, \xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}\}$. Three assets are issued, two at $\xi_0$ and one at $\xi_1$. For all non-arbitrage price $q = (q_1, q_2, q_3)$, the return matrix and the full return matrix are:

\[
V = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

and

\[
W(q) = \begin{bmatrix}
-q_1 & 0 & -q_3 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

One easily checks $\text{rank} V = \text{rank} W(q) = 3$ whatever is the price vector of assets $q$.

Remark 3. In Magill and Quinzii [3], it is assumed that a long-term asset is re-traded at each node after its issuance node. In Angeloni and Conet [1], it is shown that a financial structure with re-trading is equivalent to a financial without re-trading by considering that a re-trade is equivalent to the issuance of a new asset.

We remark that if the financial structure has long-term assets with re-trading, then Assumption R is not satisfied by the equivalent financial structure without re-trading. Let us give an example. Let us consider the date-event tree $D = \{\xi_0, \xi_1, \xi_2, \xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}\}$. Two assets are issued at $\xi_0$ with dividend processes

\[
V^1 = (0, (0,0), (1,0,1,0)) \quad V^2 = (0, (0,0), (0,1,0,1))
\]

If these two assets are re-traded at each non-terminal node successor of $\xi_0$, for all no-arbitrage price $q = (q_1(\xi_0), q_2(\xi_0), q_1(\xi_1), q_2(\xi_1), q_1(\xi_2), q_2(\xi_2))$, the full payoff matrix is:

\[
W_{MQ}(q) = \begin{bmatrix}
-q_1(\xi_0) & -q_2(\xi_0) & 0 & 0 & 0 & 0 \\
q_1(\xi_1) & q_2(\xi_1) & -q_1(\xi_1) & -q_2(\xi_1) & 0 & 0 \\
q_1(\xi_2) & q_2(\xi_2) & 0 & 0 & -q_1(\xi_2) & -q_2(\xi_2) \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
But if, following the methodology of Angeloni-Cornet [1], we consider a financial structure with 6 assets without retrading, we obtain the following full payoff matrix with \( q = (\tilde{q}_1, \tilde{q}_2, \tilde{q}_{11}, \tilde{q}_{12}, \tilde{q}_{21}, \tilde{q}_{22}) \):

\[
\mathbf{W}_{AC}(\tilde{q}) = \begin{pmatrix}
-\tilde{q}_1 & -\tilde{q}_2 & 0 & 0 & 0 & 0 \\
0 & 0 & -\tilde{q}_{11} & -\tilde{q}_{12} & 0 & 0 \\
0 & 0 & 0 & 0 & -\tilde{q}_{21} & -\tilde{q}_{22} \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 
\end{pmatrix}
\]

We remark that the two financial structure are equivalent when \( q = \tilde{q} \) since, by performing elementary operations on the columns of \( \mathbf{W}_{AC}(q) \), we obtain \( \mathbf{W}_{MQ}(q) \). Assumption R is not satisfied because the returns on assets issued nodes \( \xi_1 \) and \( \xi_2 \) are redundant with the return of assets issued to node \( \xi_0 \). As already remarked in Magill-Quinzii [3], the rank of the full payoff matrix \( \mathbf{W}_{MQ}(q) \) depends on the asset price vector \( q \).

**Proof of Proposition 2**

For all \( \xi \in \mathbb{D}^e \), we denote by \( n(\xi) \) the number of assets issued at this node and by \( \text{rk}(\xi) \) the rank of \( \mathbf{V}_{\mathcal{J}(\xi)} \). We also simplify the notation by defining \( \mathbf{V}_{\mathcal{J}(\xi)} := \mathbf{V}_{\mathcal{J}(\xi)}(q) := \mathbf{W}_{\mathcal{J}(\xi)}(q) \). We assume without any loss of generality that the columns of \( \mathbf{V} \) are ranked in such a way that the \( \text{rk}(\xi) \) first columns of \( \mathbf{V}_{\mathcal{J}(\xi)} \) are linearly independent.

**Step 1:** For all \( \xi \in \mathbb{D}^e \), \( \text{rank} \mathbf{V}_{\mathcal{J}(\xi)}(q) = \text{rk}(\xi) \).

If \( \text{rk}(\xi) = n(\xi) \), \( \text{rank} \mathbf{V}_{\mathcal{J}(\xi)}(q) = n(\xi) \). Indeed, since \( \text{rk}(\xi) = n(\xi) \), there exists a regular \( n(\xi) \) square sub-matrix of \( \mathbf{V}_{\mathcal{J}(\xi)} \). Since \( \mathbf{W}_{\mathcal{J}(\xi)}(q) \) is obtained from \( \mathbf{V}_{\mathcal{J}(\xi)}(q) \) by replacing a zero row by the row of asset prices issued at \( \xi \), the regular \( n(\xi) \) square sub-matrix is also a sub-matrix of \( \mathbf{W}_{\mathcal{J}(\xi)}(q) \), hence the rank of \( \mathbf{W}_{\mathcal{J}(\xi)}(q) \) is higher or equal to \( n(\xi) \). But, since \( n(\xi) \) is the number of columns of \( \mathbf{W}_{\mathcal{J}(\xi)}(q) \), then its rank is lower or equal to \( n(\xi) \) so that we obtain the desired result.

If \( \text{rk}(\xi) < n(\xi) \), let us consider \( \lambda = (\lambda(\xi))_{\xi \in \mathbb{D}^+} \in \mathbb{R}_{++}^n \) such that \( \text{Tr}(W, q) \lambda = 0 \). Such \( \lambda \) exists since \( q \) is a no-arbitrage price.

Let \( \{j_1, \ldots, j_{\text{rk}(\xi)}, j_{\text{rk}(\xi)+1}, \ldots, j_{n(\xi)}\} \) be the assets issued at date \( \xi \) with \( (V_{j_1}, \ldots, V_{j_{\text{rk}(\xi)}}) \) linearly independent. By the same argument as above, \( (W_{j_1}(q), \ldots, W_{j_{\text{rk}(\xi)}}(q)) \) are also linearly independent. Hence the rank of \( \mathbf{W}_{\mathcal{J}(\xi)}(q) \) is larger or equal to \( \text{rk}(\xi) \). Let us now prove by contraposition that the rank of \( \mathbf{W}_{\mathcal{J}(\xi)}(q) \) is not strictly larger than \( \text{rk}(\xi) \).

If \( \text{rank} \mathbf{W}_{\mathcal{J}(\xi)}(q) > \text{rk}(\xi) \), there exists \( \nu \geq \text{rk}(\xi) + 1 \) such that \( (W_{j_1}(q), \ldots, W_{j_{\text{rk}(\xi)}}(q)) \) are linearly independent. Since the rank of \( \mathbf{V}_{\mathcal{J}(\xi)}(q) \) is \( \text{rk}(\xi) \), \( V_{j_\nu} \) is a linear combination of \( (V_{j_1}, \ldots, V_{j_{\text{rk}(\xi)}}) \), hence there exists \( (\alpha_1, \ldots, \alpha_{\text{rk}(\xi)}) \) such that \( \sum_{k=1}^{\text{rk}(\xi)} \alpha_k V_{j_k} = V_{j_\nu} \).

\[\text{Note that we do not use the fact that the asset price is a no-arbitrage price in this part of the proof.}\]
Since \( \sum' W(q) \lambda = 0 \), \( \lambda(\xi)q_{j\nu} = \sum_{\xi' > \xi} \lambda(\xi') V^{j\nu}(\xi') \). Hence \( \lambda(\xi)q_{j\nu} \) is equal to

\[
\sum_{\xi' > \xi} \left[ \lambda(\xi') \sum_{\kappa=1}^{\text{rk}(\xi')} \alpha_{\kappa} V^{j\nu}(\xi') \right] = \sum_{\kappa=1}^{\text{rk}(\xi)} \left[ \alpha_{\kappa} \sum_{\xi' > \xi} \lambda(\xi') V^{j\nu}(\xi') \right]
\]

Hence \( q_{j\nu} = \sum_{\kappa=1}^{\text{rk}(\xi)} \alpha_{\kappa} q_{j\kappa} \), which together with \( \sum_{\kappa=1}^{\text{rk}(\xi)} \alpha_{\kappa} V^{j\nu} = V^{j\nu} \) imply that \( \sum_{\kappa=1}^{\text{rk}(\xi)} \alpha_{\kappa} W^{j\nu}(q) = W^{j\nu}(q) \). But this contradicts the fact that \( (W^{j_1}(q), \ldots, W^{j_k}(q), W^{j_q}(q)) \) are linearly independent.

For \( \kappa = 1, \ldots, k \), let \( \text{rk}_{\kappa} = \sum_{\xi \in D_{\kappa}} \text{rk}(\xi) \).

**Step 2:** \( \forall \kappa \in \{1, \ldots, k\}, \text{rank } V_{\mathcal{J}_{\kappa}} = \sum_{\xi \in D_{\kappa}} \text{rk}(\xi) = \text{rk}_{\kappa} \) and \( \text{rank } W_{\mathcal{J}_{\kappa}}(q) = \sum_{\xi \in D_{\kappa}} \text{rank } V_{\mathcal{J}(\xi)}(q) = \text{rk}_{\kappa} \)

If \( \# D_{\kappa} = 1 \), this coincides with what is proved in Step 1. If \( \# D_{\kappa} > 1 \), let \( \xi \in D_{\kappa} \). Then

\[
\left[ \sum_{\{\xi' \in D_{\kappa} \setminus \{\xi\} \}} \text{Vect}\left( V_{\mathcal{J}(\xi')} \right) \right] \cap \text{Vect}\left( V_{\mathcal{J}(\xi)} \right) = \{0\}
\]

Indeed, the return of the asset \( j \in \mathcal{J}(\xi) \) can be non zero only on the subtree \( D^+(\xi) \), whereas for the asset \( j \in \mathcal{J}(\xi') \) for \( \xi' \in D_{\kappa} \setminus \{\xi\} \), the returns on the subtree \( D^+(\xi) \) are identically equal to 0. This implies that the subspaces \( \left( \text{Vect}\left( V_{\mathcal{J}(\xi')} \right) \right)_{\xi \in D_{\kappa}} \) are in direct sum so, using Step 1, we get the following formula for the dimensions:

\[
\text{dim } \text{Vect} \left( V_{\mathcal{J}_{\kappa}} \right) = \sum_{\xi \in D_{\kappa}} \text{dim } \text{Vect} \left( V_{\mathcal{J}(\xi)} \right) = \sum_{\xi \in D_{\kappa}} \text{rk}(\xi) = \text{rk}_{\kappa}
\]

For the matrix \( W(q) \), the proof is the same as above if we remark that the full return of an asset \( j \in \mathcal{J}(\xi) \) can be non zero only on the subtree \( \xi \cup D^+(\xi) \).

Hence if \( \xi \) and \( \xi' \) are two different emission nodes in \( D_{\kappa} \), there is no node \( \xi'' \) such that the coordinates of a column vectors of the matrix \( W_{\mathcal{J}(\xi)}(q) \) and of a column vector of the matrix \( W_{\mathcal{J}(\xi')} \) are both non zero. Hence we get the following formula from which the result is a direct consequence of Step 1:

\[
\left[ \sum_{\{\xi' \in D_{\kappa} \setminus \{\xi\} \}} \text{Vect}\left( W_{\mathcal{J}(\xi')}(q) \right) \right] \cap \text{Vect}\left( W_{\mathcal{J}(\xi)}(q) \right) = \{0\}
\]

**Step 3:** \( \text{rank } V = \sum_{\kappa=1}^{k} \text{rank } V_{\mathcal{J}_{\kappa}} = \sum_{\kappa=1}^{k} \text{rk}_{\kappa} \) and \( \text{rank } W(q) = \sum_{\kappa=1}^{k} \text{rank } W_{\mathcal{J}_{\kappa}}(q) = \sum_{\kappa=1}^{k} \text{rk}_{\kappa} \).
We first remark that $\text{Vect} (V) = \bigoplus_{k=1}^{k} \text{Vect} (V_{\mathcal{J}^k})$ which implies using Step 2 that $\text{rk} V \leq \sum_{k \in \{1, \ldots, k\}} \text{rk} V_{\mathcal{J}^k} = \sum_{k=1}^{k} \text{rk} _k$.

We remark that if $k = 1$, then the result is obvious. If $k > 1$, we will prove that the rank of $V$ is equal to $\sum_{k=1}^{k} \text{rk} _k$ by showing that a family of column vectors of $V$ of cardinal $\sum_{k=1}^{k} \text{rk} _k$ is linearly independent.

For all $\xi \in \mathbb{D}^{c}$, let $\mathcal{J}'(\xi) \subset \mathcal{J}(\xi)$ such that $\# \mathcal{J}'(\xi) = \text{rk} (\xi)$ and the family $(V_j)_{j \in \mathcal{J}'(\xi)}$ is linearly independent. For all $\kappa = 1, \ldots, k$, $\mathcal{J}'_\kappa = \cup_{\xi \in \mathbb{D}^{c}} \mathcal{J}'(\xi)$ and $\mathcal{J}' = \cup_{\kappa=1}^{k} \mathcal{J}'_\kappa$. We now prove that the family $(V_j)_{j \in \mathcal{J}'}$ is linearly independent.

Let $(\alpha_j) \in \mathbb{R}^{\mathcal{J}'}$ such that $\sum_{j \in \mathcal{J}'} \alpha_j V_j = 0$. We work by backward induction on $\kappa$.

For all $\xi \in \mathbb{D}^{c}$, $\sum_{j \in \mathcal{J}'} \alpha_j V_{D^+(\xi)} = 0$. For all $j$ such that $\xi (j) \notin \mathbb{D}^{-}(\xi) \cup \{\xi\}$, $V_{D^+(\xi)} = 0$. So, one gets

$$\sum_{j \in \mathcal{J}'(\xi)} \alpha_j V_{D^+(\xi)} + \sum_{\xi' \in \mathbb{D}^{-}(\xi)} \sum_{j \in \mathcal{J}'(\xi')} \alpha_j V_{D^+(\xi')} = 0$$

From Assumption R,

$$\text{Vect} \left( V_{D^+(\xi)}' \right) \cap \text{Vect} \left( V_{D^+(\xi)} \right) = \{0\} .$$

From the above equality,

$$\sum_{j \in \mathcal{J}'(\xi)} \alpha_j V_{D^+(\xi)} \in \text{Vect} \left( V_{D^+(\xi)}' \right) \cap \text{Vect} \left( V_{D^+(\xi)} \right)$$

hence $\sum_{j \in \mathcal{J}'(\xi)} \alpha_j V_{D^+(\xi)} = 0$.

By construction, the family $(V_j)_{j \in \mathcal{J}'(\xi)}$ is linearly independent and for all $\xi' \notin \mathbb{D}^+(\xi)$, $V_j (\xi') = 0$, so the family $(V_j)_{j \in \mathcal{J}'(\xi)}$ is linearly independent. Hence, from above, one deduces that $\alpha_j = 0$ for all $j \in \mathcal{J}'(\xi)$.

Hence, for all $j \in \mathcal{J}'_\kappa$, $\alpha_j = 0$.

If $k = 1$, we are done since we have prove in Step 2 that the subspaces $(\text{Vect} (V_{\mathcal{J}^k}(\xi)))_{\xi \in \mathbb{D}^{c}}$ are in direct sum, so the family $(V_j)_{j \in \mathcal{J}^k}$ is linearly independent, hence for all $j \in \mathcal{J}^k$, $\alpha_j = 0$.

If $k > 1$, we do again the same argument as above to prove that Indeed, since we already know that for all $j \in \mathcal{J}'_k$, $\alpha_j = 0$, for all $\xi \in \mathbb{D}^{c}_{\tau - 1}$, for all $j \in \mathcal{J}'_k$, $\alpha_j V_{D^+(\xi)} = 0$, hence $\sum_{j \in \mathcal{J}'} \alpha_j V_{D^+(\xi)} = 0$ implies

$$\sum_{j \in \mathcal{J}'(\xi)} \alpha_j V_{D^+(\xi)} + \sum_{\xi' \in \mathbb{D}^{-}(\xi)} \sum_{j \in \mathcal{J}'(\xi')} \alpha_j V_{D^+(\xi')} = 0 .$$

Using again Assumption R, one then deduces that for all $j \in \mathcal{J}'_{\kappa-1}$, $\alpha_j = 0$.

Consequently, after a finite number of steps, we show that all $\alpha_j$ are equal to 0, which implies that the family $(V_j)_{j \in \mathcal{J}'}$ is linearly independent.
For the second part concerning the matrix \( W(q) \) the proof is identical since for all \( \xi \in \mathbb{D}^e \), the family \( \{W^j(\xi)\}_{j \in \mathcal{J}(\xi)} \) is linearly independent and for all \( j \in \mathcal{J}'(\xi) \cup \left( \bigcup_{\zeta \in \mathbb{D}^{\bar{J}-} \left( \xi \right)} J(\zeta) \right) \), \( V^j_{\mathcal{D}+}(\xi) = W^j_{\mathcal{D}+}(\xi)(q) \).

**Remark 4.** If the price \( q \) exhibits an arbitrage, then even under Assumption R, the rank of \( V \) can be not equal to the rank of \( W(q) \). With a three dates economy where \( \mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}\} \), two assets issued at \( \xi_0 \) and one asset issued at \( \xi_1 \), the asset price \( q = (1, \frac{3}{2}, 1) \), then

\[
V = \begin{bmatrix}
0 & 0 & 0 \\
1 & 2 & 0 \\
1 & 2 & 0 \\
0 & 0 & 1 \\
1 & 2 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \quad \text{and} \quad W(q) = \begin{bmatrix}
-1 & -\frac{3}{2} & 0 \\
1 & 2 & -1 \\
1 & 2 & 0 \\
1 & 2 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

We note that \( \text{rank} V = 2 < \text{rank} W(q) = 3 \). Nevertheless, the following result shows that if the return vectors are not redundant at each node, then the equality of ranks holds true even with an arbitrage price.

**Proposition 3.** Let us assume that \( V \) satisfies Assumption R.

1) For all price \( q \in \mathbb{R}^j \), \( \text{rank} V \leq \text{rank} W(q) \).

2) Furthermore, if for all \( \xi \in \mathbb{D}^e \), \( \text{rank} V^j(\xi) = n(\xi) \), the number of assets issued at this node, then \( \text{rank} V = \text{rank} W(q) \) for all price \( q \in \mathbb{R}^j \).

**Proof.** 1) The proof is just an adaptation of the proof of Proposition 2. In the first step, since the price \( q \) is not supposed to be a non-arbitrage price, we get \( \text{rank} W^j(\xi)(q) \geq \text{rk}(\xi) \) instead of an equality. For the two next steps, the proofs never uses the fact that \( q \) is a non-arbitrage price, so we can replicate them to obtain \( \text{rank} W(q) \geq \sum_{\xi \in \mathbb{D}^e} \text{rk}(\xi) = \text{rank} V \).

2) If \( \text{rk}(\xi) = n(\xi) \) for all \( \xi \), then \( \sum_{\xi \in \mathbb{D}^e} \text{rk}(\xi) \) is the cardinal of \( \mathcal{J} \), which is the number of column of the matrix \( W(q) \). So \( \text{rank} W(q) \leq \sum_{\xi \in \mathbb{D}^e} \text{rk}(\xi) = \text{rank} V \). \( \Box \)

If the assets issued at each node are linearly independent, then we can also obtain the equality of the rank under a slightly weaker assumption than Assumption R where we only deal with the returns at the immediate successors of a node \( \xi \) instead of looking at the whole returns for all successors.

**Corollary 1.** Let us assume that:

1) \( \forall \xi \in \mathbb{D}^e \), \( \text{rank} V^j_{\mathcal{D}+}(\xi) = n(\xi) \)

and

2) \( \text{Vect} \left( V^j_{\mathcal{D}+}(\xi) \right) \cap \text{Vect} \left( V^j_{\mathcal{D}+}(\xi^{-1}) \right) = \{0\} \).

Then \( \text{rank} V = \text{rank} W(q) \).

**Proof.** We show that the assumptions of Proposition 3 are satisfied. First, we remark that \( V^j_{\mathcal{D}+}(\xi) \) is a sub-matrix of \( V^j(\xi) \), so \( n(\xi) = \text{rank} V^j_{\mathcal{D}+}(\xi) \leq \text{rank} V^j(\xi) \).
4 Existence of equilibrium

4.1 The stochastic exchange economy

At each node \( \xi \in \mathbb{D} \), there is a spot market where a finite set \( \mathbb{H} = \{1, \ldots, H\} \) of divisible and physical goods available. We assume that each good life does not have more than one date. In this model, a product is a pair \((h, \xi)\) of a physical good \( h \in \mathbb{H} \) and node \( \xi \in \mathbb{D} \) which will be available, then the commodity space is \( \mathbb{R}^L \), where \( L = \mathbb{H} \times \mathbb{D} \) is called a consumer, that is to say \( x = (x(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^L \), where \( x(\xi) = (x(h, \xi))_{h \in \mathbb{H}} \in \mathbb{R}^\mathbb{H} \) for each \( \xi \in \mathbb{D} \).

We denote by \( p = (p(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^L \) the vector of spot prices and \( p(\xi) = (p(h, \xi))_{h \in \mathbb{H}} \in \mathbb{R}^\mathbb{H} \) is called the spot price at node \( \xi \). The spot price \( p(h, \xi) \) is the price at the node \( \xi \) for immediate delivery of one unit of physical good \( h \). Thus the value of a consumption \( x(\xi) \) to node \( \xi \in \mathbb{D} \) (measured in unit account of the node \( \xi \)) is

\[
p(\xi) \cdot x(\xi) = \sum_{h \in \mathbb{H}} p(h, \xi) x(h, \xi)
\]

Each agent \( i \in I \) has a set of \( X_i \subset \mathbb{R}^L \) which consists of all its possible consumption. An allocation is an element \( x \in \prod_{i \in I} X_i \) and we denote by \( x_i \) by the consumption of agent \( i \), which is the projection of \( x \) on \( X_i \).

The tastes of each consumer \( i \in I \) are represented by a strict preference correspondence \( P_i : \prod_{j \in J} X_j \rightarrow X_i \), where \( P_i(x) \) defines the set of consumption that are strictly preferred to \( x_i \) for agent \( i \), given the consumption \( x_j \) for the other consumers \( j \neq i \). \( P_i \) represents the consumer tastes, but also his behavior with respect to time and uncertainty, especially his impatience and attitude to risk. If consumer preferences are represented by utility functions \( u_i : X_i \rightarrow \mathbb{R} \) for each \( i \in I \), the strict preference correspondence is defined by \( P_i(x) = \{ x_i \in X_i \mid u_i(x_i) > u_i(x_i) \} \).
Finally, for each node $\xi \in \mathcal{D}$, every consumer $i \in \mathcal{I}$ has a node endowment $e_i(\xi) \in \mathbb{R}^H$ (contingent on the fact that $\xi$ prevails) and we denote by $e_i = (e_i(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}^{\mathcal{J}}$ the endowments for the whole set of nodes. The exchange economy $\Sigma$ can be summarized by

$$\Sigma = \left[\mathbb{D}, \mathbb{H}, \mathcal{I}, (X_i, P_i, e_i)_{i \in \mathcal{I}}\right].$$

### 4.2 Financial equilibrium

We now consider a financial exchange economy, which is defined as the couple of an exchange economy $\Sigma$ and a financial structure $\mathcal{F}$. It can thus be summarized by

$$\left(\Sigma, \mathcal{F}\right) := \left[\mathbb{D}, \mathbb{H}, \mathcal{I}, (X_i, P_i, e_i)_{i \in \mathcal{I}}, \mathcal{J}, (Z_i)_{i \in \mathcal{I}}, (\xi (j))_{j \in \mathcal{J}}, V\right].$$

Given the price $(p, q) \in \mathbb{R}^I \times \mathbb{R}^J$, the budget set of consumer $i \in \mathcal{I}$ is $B^i_{\mathcal{F}}(p, q)$ defined by:

$$\{(x_i, z_i) \in X_i \times Z_i : \forall \xi \in \mathcal{D}, p(\xi) \bullet_{\mathcal{H}} [x_i(\xi) - e_i(\xi)] \leq |W_{\mathcal{F}}(p, q) z_i| (\xi)\}$$

or

$$\{(x_i, z_i) \in X_i \times Z_i : p \circ (x_i - e_i) \leq W_{\mathcal{F}}(p, q) z_i\}.$$

We now introduce the equilibrium notion

**Definition 1.** An equilibrium of the financial exchange economy $(\Sigma, \mathcal{F})$ is a list of strategies and prices $(\bar{x}, \bar{z}, \bar{p}, \bar{q}) \in (\mathbb{R}^H)^I \times \prod_{i \in \mathcal{I}} Z_i \times \mathbb{R}^I \setminus \{0\} \times \mathbb{R}^J$ such that

(a) for every $i \in \mathcal{I}$, $(\bar{x}_i, \bar{z}_i)$ maximizes the preferences $P^i$ in the budget set $B^i_{\mathcal{F}}(\bar{p}, \bar{q})$, in the sense that

$$(x_i, z_i) \in B^i_{\mathcal{F}}(\bar{p}, \bar{q}) \text{ and } [P^i(\bar{x}) \times Z_i] \cap B^i_{\mathcal{F}}(\bar{p}, \bar{q}) = \emptyset;$$

(b) $\sum_{i \in \mathcal{I}} \bar{x}_i = \sum_{i \in \mathcal{I}} e^i$ and $\sum_{i \in \mathcal{I}} \bar{z}_i = 0$.

### 4.3 No-arbitrage and financial equilibrium

Angeloni-Cornet[1] noted that when portfolios may be constrained, the concept of no-arbitrage has to be suitably modified. In particular, we shall make a distinction between the definitions of arbitrage-free portfolio and arbitrage-free financial structure.

**Definition 2.** Given the financial structure $\mathcal{F} = \left(\mathcal{J}, (Z_i)_{i \in \mathcal{I}}, (\xi (j))_{j \in \mathcal{J}}, V\right)$, the portfolio $z_i \in Z_i$ is said with no arbitrage opportunities or to be arbitrage-free for agent $i \in \mathcal{I}$ at the price $(p, q) \in \mathbb{R}^I \times \mathbb{R}^J$ if there is no portfolio $z_i \in Z_i$ such that $W_{\mathcal{F}}(p, q) z_i > W_{\mathcal{F}}(p, q) (\bar{z}_i)$, that is, $[W_{\mathcal{F}}(p, q) z_i] (\xi) \geq [W_{\mathcal{F}}(p, q) (\bar{z}_i)] (\xi)$, for every $\xi \in \mathcal{D}$, with at least one strict inequality, or, equivalently, if:

$$W_{\mathcal{F}}(p, q) (Z_i - z_i) \cap \mathbb{R}^I_+ = \{0\}.$$

---

3For $x = (x(\xi))_{\xi \in \mathcal{D}}, p = (p(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}^E = \mathbb{R}^{\mathcal{H} \times \mathcal{D}}$ (with $x(\xi), p(\xi) \in \mathbb{R}^H$) we let $p \circ x = (p(\xi) \bullet_{\mathcal{H}} x(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}^D$. 

The financial structure is said to be arbitrage-free at \((p, q)\) if there exists no portfolio \((z_i) \in \prod_{i \in \mathcal{I}} Z_i\) such that \(W_F(p, q) \left( \sum_{i \in \mathcal{I}} z_i \right) > 0\), or, equivalently, if:

\[
W_F(p, q) \left( \sum_{i \in \mathcal{I}} Z_i \right) \cap \mathbb{R}_+^D = \{0\}.
\]

Let the financial structure \(\mathcal{F}\) be arbitrage-free at \((p, q)\), and let \((\bar{z}_i) \in \prod_{i \in \mathcal{I}} Z_i\) such that \(\sum_{i \in \mathcal{I}} \bar{z}_i = 0\). Then, for every \(i \in \mathcal{I}\), \(\bar{z}_i\) is arbitrage-free at \((p, q)\). The converse is true, for example, when some agent’s portfolio set is unconstrained, that is, when \(Z_i = \mathbb{R}^d\) for some \(i \in \mathcal{I}\).

We recall that equilibrium portfolios are arbitrage-free under the following non-satiation assumption:

**Assumption NS**

(i) (Non-Saturation at Every Node.) For every \(\bar{x} \in \prod_{i \in \mathcal{I}} X_i\) such that \(\sum_{i \in \mathcal{I}} \bar{x}_i = \sum_{i \in \mathcal{I}} e_i\), for every \(\xi \in \mathbb{D}\), there exists \(x \in \prod_{i \in \mathcal{I}} X_i\) such that, for each \(\xi' \neq \xi\), \(x_i(\xi') = \bar{x}(\xi')\) and \(x_i \in P^i(\bar{x})\);

(ii) if \(x_i \in P^i(\bar{x})\), then \([\bar{x}_i, x_i] \subset P^i(\bar{x})\).

**Proposition 4.** Under Assumption (NS), if \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) is an equilibrium of the economy \((\mathcal{S}, \mathcal{F})\), then \(\bar{z}_i\) is no-arbitrage at price \((\bar{p}, \bar{q})\) for every \(i \in \mathcal{I}\).

The proof is given in Angeloni-Cornet [1] as well as the following characterization.

**Theorem 1.** Let \(\mathcal{F} = \left(\mathcal{J}, (Z_i)_{i \in \mathcal{I}}, (\xi(j))_{j \in \mathcal{J}}, V\right)\), let \((p, q) \in \mathbb{R}^1 \times \mathbb{R}^q\), for \(i \in \mathcal{I}\), let \(z_i \in Z_i\), assume that \(Z_i\) is convex and consider the following statements:

(i) There exists \(\lambda^i = (\lambda^i(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}_{++}^D\) such that \(\lambda^i(\xi) \in N_{Z_i}(z_i)^4\), or, equivalently, there exists \(\eta \in N_{Z_i}(z_i)\) such that:

\[
\lambda^i(\xi(j)) \eta_j = \sum_{\xi > \xi(j)} \lambda^i(\xi) V^j(p, \xi) - \eta_j \text{ for every } j \in \mathcal{J}
\]

(ii) the portfolio \(z_i\) is no-arbitrage-free for agent \(i \in \mathcal{I}\) at price \((p, q)\).

The implication \([\text{(i)} \Rightarrow \text{(ii)}]\) always holds and the converse is true under the additional assumption that \(Z_i\) is a polyhedral\(^5\) set.

### 4.4 Existence of equilibrium

We introduce the following assumptions on the consumers and the financial structure. They are borrowed from Angeloni-Cornet [1] and Cornet-Gopalan [2]:

\(^4\)we recall that \(N_{Z_i}(z_i)\) is the normal cone to \(Z_i\) at \(z_i\), which is defined as \(N_{Z_i}(z_i) = \{\eta \in \mathbb{R}^d : \eta \cdot z_i \geq \eta \cdot z_i' \forall z_i' \in Z_i\}\)

\(^5\)A subset \(C \subset \mathbb{R}^n\) is said to be polyhedral if it is the intersection of finitely many closed half-spaces, namely \(C = \{x \in \mathbb{R}^n : Ax \leq b\}\), where \(A\) is a real \((m \times n)\)-matrix , and \(b \in \mathbb{R}^m\). Note that polyhedral sets are always closed and convex and that the empty set and the whole space \(\mathbb{R}^n\) are both polyhedral.
Proposition 5. (a) Let we get a result for all state prices. with assumptions on the matrix $V$, regardless of the price of no-arbitrage. Then and rank $V$ for all $(p, q, \eta)$ the preference correspondence (ii) open set $V$ (Irreflexivity) $\bar{x} / \in X_i$ is lower semicontinuous $^6$ and $P^i(\bar{x})$ is convex;

(iii) for every $x_i \in P^i(\bar{x})$ for every $x'_i \in X_i$, $x'_i \neq x_i$, $[x'_i, x_i] \cap P^i(\bar{x}) \neq \emptyset$;

(iv) (Irreflexivity) $\bar{x} \notin P^i(\bar{x})$;

(v) (Non-Saturation of Preferences at Every Node) if $\sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i$, for every $\xi \in \mathbb{D}$ there exists $x \in \prod_{i \in I} X_i$ such that, for each $\xi' \neq \xi$, $x_i(\xi') = x_i(\xi)$ and $x_i \in P^i(\bar{x})$;

(vi) (Strong Survival Assumption) $e_i \in intX_i$.

Note that these assumptions are satisfied on $P^i$ particularly when agents’ preferences are given by a utility that is continuous, strongly monotone and quasi-concave.

**Assumption (C) (Consumption Side)** For all $i \in I$ and all $\bar{x} \in \prod_{i \in I} X_i,$

(i) $X_i$ is a closed and convex subset of $\mathbb{R}^2$;

(ii) the preference correspondence $P^i$, from $\prod_{i \in I} X_i \to X_i$, is lower semicontinuous$^6$ and $P^i(\bar{x})$ is convex;

(iii) for every $x_i \in P^i(\bar{x})$ for every $x'_i \in X_i$, $x'_i \neq x_i$, $[x'_i, x_i] \cap P^i(\bar{x}) \neq \emptyset$;

(iv) (Irreflexivity) $\bar{x} \notin P^i(\bar{x})$;

(v) (Non-Saturation of Preferences at Every Node) if $\sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i$, for every $\xi \in \mathbb{D}$ there exists $x \in \prod_{i \in I} X_i$ such that, for each $\xi' \neq \xi$, $x_i(\xi') = x_i(\xi)$ and $x_i \in P^i(\bar{x})$;

(vi) (Strong Survival Assumption) $e_i \in intX_i$.

In Angeloni-Cornet [1], the result on the existence of equilibrium requires that the set $B(\lambda)$ of admissible consumptions and portfolios for a given state price $\lambda \in \mathbb{R}^3_{+}$, that is,

$$\{(x, z) \in \prod_{i \in I} X_i \times \prod_{i \in I} Z_i : \exists (p, q) \in B_{L_i} (0, 1) \times \mathbb{R}^J, \quad W^i (p, q) \lambda \in B_{J_i} (0, 1), (x_i, z_i) \in B^i (p, q) \text{ for every } i \in I, \quad \sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i, \quad \sum_{i \in I} z_i = 0\}.$$ 

is bounded. It is proved that this holds true if the assets are all short-term and rank $V = \# \mathcal{J}$ or, if there are long-term assets, that rank $W^i (p, q) \lambda = \# \mathcal{J}$ for all $(p, q, \eta) \in B_L (0, 1) \times \mathbb{R}^J \times B_J (0, 1)$ such that $W^i (p, q) \lambda = \eta$.

Our contribution in the next proposition is to obtain an existence result only with assumptions on the matrix $V$, regardless of the price of no-arbitrage. Then we get a result for all state prices.

**Proposition 5. (a)** Let

$$(\Sigma, \mathcal{F}) := \left[\mathbb{D}, \mathbb{H}, \mathcal{I}, (X_i, P^i, e_i)_{i \in I}, \mathcal{J}, (Z_i)_{i \in I}, (\xi (j))_{j \in \mathcal{J}}, V\right]$$

be a financial economy with nominal assets satisfying Assumptions (C), (F), (R) and rank$(V) = \# \mathcal{J}$. Let $i_0 \in I$ be some agent such that

$^6$A correspondence $\phi : X \to Y$ is said lower semicontinuous at $x_0 \in X$ if, for every open set $V \subset Y$ such that $V \cap \phi(x_0)$ is nonempty, there exists a neighborhood $U$ of $x_0$ in $X$ such that, for all $x \in U$, $V \cap \phi(x)$ is nonempty. The correspondence $\phi$ is said to be lower semicontinuous if it is lower semicontinuous at each point of $X$.

$^7$This is satisfied, in particular, when $P^i(\bar{x})$ is open in $X_i$ (for its relative topology).
4 EXISTENCE OF EQUILIBRIUM

\[ 0 \in \text{int } Z_{i_0}. \] Then, for any given \( \lambda \in \mathbb{R}^D_{++}, \) there exists an equilibrium \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) of \((\Sigma, F)\) such that, for every \( \xi \in \mathcal{D}, \bar{p}(\xi) \neq 0 \) and

\[ ^1W_F(\bar{q}) \lambda \in N_{Z_{i_0}}(\bar{z}_{i_0}), \]

or, equivalently, there exists \( \bar{\eta} \in N_{Z_{i_0}}(\bar{z}_{i_0}) \) such that

\[ \lambda(\xi(j)) \bar{q}_j = \sum_{\xi > \xi(j)} \lambda(\xi) V^j_\xi - \bar{\eta}_j \text{ for every } j \in \mathcal{J}. \]

\( \text{(b)} \) If moreover \( \bar{z}_{i_0} \in \text{int } Z_{i_0}, \) then \(^1W_F(\bar{q}) \lambda = 0, \) or, equivalently,

\[ \lambda(\xi(j)) \bar{q}_j = \sum_{\xi > \xi(j)} \lambda(\xi) V^j_\xi \text{ for every } j \in \mathcal{J}, \]

hence the financial structure \( F \) is arbitrage-free at \((\bar{p}, \bar{q})\).

This proposition is a direct consequence of Theorem 3.1 of Angeloni-Cornet [1] and the following proposition.

**Proposition 6.** Let \((\Sigma, F) := [\mathcal{D}, H, I, (X_i, P_i, e_i)_{i \in I}, \mathcal{J}, (Z_i)_{i \in I}, (\xi(j))_{j \in \mathcal{J}}, V]\) be a financial economy satisfying for all \( i \in I, X_i \) is lower bounded, \( F \) consists of nominal assets and satisfies the hypothesis \((F)\). We assume that Assumption \( R \) is satisfied and that the rank \( V = \# \mathcal{J}. \) Then, for any given \( \lambda \in \mathbb{R}^D_{++}, \) \( B(\lambda) \) is bounded.

**Remark 5.**

i) The conditions given in the above proposition for the boundedness of \( B(\lambda) \) are independent of the asset prices \( q \) which is an improvement with respect to the paper (Proposition 3.1) of Angeloni-Cornet [1].

ii) If the assets are all short term, Assumption \( R \) holds true (See Proposition 1 (i)) and therefore Proposition 6 generalizes Proposition 3.1 (iii) of Angeloni-Cornet.

iii) If one of the conditions of Proposition 1 is satisfied, then the result holds since Assumption \( R \) is satisfied.

The proof of Proposition 6 is in Appendix.

In Proposition 5, there is no restriction on the portfolio sets beyond Assumption \( F, \) but we encompass only the case without redundant assets, that is that we have the assumption \( \text{rank}(V) = \# \mathcal{J}. \) In the following corollary, we no more assume the non-redundancy of the assets but we have unconstrained portfolio sets of the agents, that is \( Z_i = \mathbb{R}^J. \)

**Proposition 7.** Let \((\Sigma, F) := [\mathcal{D}, H, I, (X_i, P_i, e_i)_{i \in I}, \mathcal{J}, (Z_i)_{i \in I}, (\xi(j))_{j \in \mathcal{J}}, V]\) be a financial economy with nominal assets satisfying Assumptions \((C), (R)\) and for every \( i \in I, Z_i = \mathbb{R}^J. \) Then, for fixed \( \lambda \in \mathbb{R}^D_{++}, \) there exists an equilibrium \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) of \((\Sigma, F)\) such that, for every \( \xi \in \mathcal{D}, \bar{p}(\xi) \neq 0 \) and

\[ ^1W_F(\bar{q}) \lambda = 0, \]

or, equivalently,

\[ \lambda(\xi(j)) \bar{q}_j = \sum_{\xi > \xi(j)} \lambda(\xi) V^j_\xi \text{ for every } j \in \mathcal{J}. \]
We remark that $B(\lambda)$ may be not bounded under the assumptions of the above proposition. Let us consider

$$(\Sigma, \mathcal{F}) := [\mathcal{D}, \mathcal{H}, \mathcal{I}, (X_i, P^i, e_i)_{i \in \mathcal{I}}, \mathcal{J}, (\xi(j))_{j \in \mathcal{J}^\prime}, V]$$

a financial exchange economy with 3 nodes without uncertainty, satisfying assumptions $C$ with $X_i = \mathbb{R}^3$, $\mathcal{I} = 2$, $H$ is a singleton, $Z_i = \mathbb{R}^3$ and such that for every asset price $q = (q_1, q_2, q_3)$ the full payoff matrix is:

$$W(q) = \begin{bmatrix} -q_1 & 0 & 0 \\ 1 & -q_2 & -q_3 \\ 2 & 1 & 2 \end{bmatrix}$$

Let $(z^n_i)$ be a sequence of elements of $\mathbb{R}^3$ such that

$$z^n_i = \begin{cases} 1 & i \neq n \\ -2n & i = n \end{cases}$$

Let us consider the no-arbitrage price $q = (2, 1, 1)$. The spot price is $p = (1, 1, 1)$. Let $e_1 = (3, 3, 3) = e_2$ and $x_1 = (2, 4, 5)$ and $x_2 = (4, 2, 1)$. Clearly, for all $n$, $(\bar{x}, z^n) \in B(\lambda)$ for $\lambda = \{1, 1, \frac{1}{2}\}$ since for all $n \in \mathbb{N}$,

$$[W_{\mathcal{F}}(p, q) z^n] = (-1, 1, 2) \text{ and } [W_{\mathcal{F}}(p, q) z^n] = (1, -1, -2).$$

Hence $B(\lambda)$ is not bounded since $(z^n)$ is not bounded.

The proof of Proposition 7 is based on the following result, which shows that the existence of an equilibrium for an exchange economy $\Sigma$ with a reduced financial structure $\mathcal{F}^\prime$ satisfying some assumptions implies the existence of equilibrium for the financial economy $(\Sigma, \mathcal{F})$ of Proposition 7.

**Proposition 8.** Let $(\Sigma, \mathcal{F})$ be the financial economy of the Proposition 7. Let $(\Sigma, \mathcal{F}^\prime) = \left[\mathcal{D}, \mathcal{H}, \mathcal{I}, (X_i, P^i, e_i)_{i \in \mathcal{I}}, \mathcal{J}^\prime, \left(\mathbb{R}^{\mathcal{J}^\prime}\right)^{\mathcal{I}}, (\xi(j))_{j \in \mathcal{J}^\prime}, V\right]$ be a financial economy satisfying $\mathcal{J}^\prime \subset \mathcal{J}$, $\# \mathcal{J}^\prime = \text{rank} V$, the columns $(V^\prime(j))_{j \in \mathcal{J}^\prime}$ are independent and $Z_i^\prime = \mathbb{R}^{\mathcal{J}^\prime}$ for every $i \in \mathcal{I}$. If $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium of financial economy $(\Sigma, \mathcal{F}^\prime)$ associated to the state price $\lambda$, then there exists $(\bar{z}, \bar{q})$ such that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium of the financial economy $(\Sigma, \mathcal{F})$ associated to the same state price $\lambda$.

The proof of Proposition 8 is in Appendix.

**Proof of Proposition 7.** Let $\mathcal{F}^\prime = \left(\mathcal{J}^\prime, (Z_i^\prime)_{i \in \mathcal{I}}, (\xi(j))_{j \in \mathcal{J}^\prime}, V^\prime\right)$ a new financial structure satisfying $\mathcal{J}^\prime \subset \mathcal{J}$, $\# \mathcal{J}^\prime = \text{rank} V$, the columns $(V^\prime(j))_{j \in \mathcal{J}^\prime}$ are independent and $Z_i^\prime = \mathbb{R}^{\mathcal{J}^\prime}$ for every $i \in \mathcal{I}$. In view of Lemma 1, it is clear that the financial economy $(\Sigma, \mathcal{F}^\prime)$ satisfies all Assumptions of Proposition 5. So there exists an equilibrium $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ of $(\Sigma, \mathcal{F}^\prime)$ such that, for every $\xi \in \mathcal{D}$, $p(\xi) \neq 0$ and

$$W_{\mathcal{F}^\prime}(\bar{q}) \lambda = 0,$$

or, equivalently,

$$\lambda(\xi(j)) q_j^\prime = \sum_{\xi > \xi(j)} \lambda(\xi) V^{\xi j}_{\xi^\prime} \text{ for every } j \in \mathcal{J}^\prime.$$
Therefore, one deduces from Proposition 8 that there exists an equilibrium 
\((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) of \((\Sigma, \mathcal{F})\) such that, for every \(\xi \in \mathcal{D}\), \(\bar{p}(\xi) \neq 0\) and 
\[W_{\mathcal{F}}(\bar{q}) \lambda = 0,\]
or, equivalently,
\[\lambda(\xi(j)) q_j = \sum_{\xi > \xi(j)} \lambda(\xi) V^j_\xi \] for every \(j \in \mathcal{J}\).
5 Appendix

For every \( i \in \mathcal{I} \), \( \lambda \in \mathbb{R}_{++}^n \), we let \( \hat{X}_i(\lambda) \) and \( \hat{Z}_i(\lambda) \) be the projections of \( B(\lambda) \) on \( X_i \) and \( Z_i \), respectively, that is:

\[
\hat{X}_i(\lambda) := \left\{ x_i \in X_i : \exists (x_j)_{j \neq i} \in \prod_{j \neq i} X_j, \exists z \in \prod_{i \in \mathcal{I}} Z_i, (x, z) \in B(\lambda) \right\}
\]

\[
\hat{Z}_i(\lambda) := \left\{ z_i \in Z_i : \exists (z_j)_{j \neq i} \in \prod_{j \neq i} Z_j, \exists x \in \prod_{i \in \mathcal{I}} X_i, (x, z) \in B(\lambda) \right\}.
\]

**Proof of Lemma 1.** Let us denote by \( k \) [resp. \( k' \)] the number of dates where there are emissions of at least one asset for the financial structure \( \mathcal{F} \) [resp. \( \mathcal{F}' \)]. It is clear that \( k' \leq k \).

By Assumption \( R \), we have: for all \( i \in \{2, \ldots, k\} \) and for all \( \xi \in \mathbb{D}_{r_i}^e \),

\[
\text{Vect}(\mathbb{D}^e_{r_i}(\mathcal{J}'(\xi^{-}))) \cap \text{Vect}(\mathbb{D}^e_{r_i}(\mathcal{J}(\xi))) = \{0\}.
\]

Let us argue by contradiction. If there exists \( i \in \{2, \ldots, k'\} \), \( \xi \in \mathbb{D}_{r_i}^e \) et \( y_i^\xi \neq 0 \) such that

\[
y_i^\xi \in \text{Vect}(\mathbb{D}^e_{r_i}(\mathcal{J}'(\xi^{-}))) \cap \text{Vect}(\mathbb{D}^e_{r_i}(\mathcal{J}(\xi)))
\]

Since \( \mathcal{J}' \subset \mathcal{J} \), \( \text{Vect}(\mathbb{D}^e_{r_i}(\mathcal{J'}(\xi^{-}))) \subset \text{Vect}(\mathbb{D}^e_{r_i}(\mathcal{J}(\xi^{-}))) \) and \( \text{Vect}(\mathbb{D}^e_{r_i}(\mathcal{J}'(\xi))) \subset \text{Vect}(\mathbb{D}^e_{r_i}(\mathcal{J}(\xi))) \). So \( y_i^\xi \in \text{Vect}(\mathbb{D}^e_{r_i}(\mathcal{J}(\xi^{-}))) \cap \text{Vect}(\mathbb{D}^e_{r_i}(\mathcal{J}(\xi))) \), which is in contradiction with the fact that \( \mathcal{F} \) satisfies Assumption \( R \). \( \square \)

**Lemma 2.** Let \( A \) be a compact subset of \( \mathbb{R}^n \). For all \( \alpha \in A \), let \( W(\alpha) : \mathbb{R}^J \to \mathbb{R}^b \) be a linear mapping. If the application \( \alpha \to W(\alpha) \) is continuous and \( \text{rank } W(\alpha) = \#J, \text{ for all } \alpha \in A \), then there exists \( c > 0 \) such that:

\[
\|W(\alpha)z\| \geq c\|z\| \text{ for every } \alpha \in A \text{ and } z \in \mathbb{R}^J
\]

For the proof, see lemma 5.1 of Angeloni-Cornet [1].

**Proof of Proposition 6** Let \( \lambda \in \mathbb{R}_{++}^n \) be fixed. We show first that for each \( i \in \mathcal{I} \) the set \( \hat{X}_i(\lambda) \) is bounded. Indeed, since the sets \( X_i \) are bounded below, there exists \( x'_j \in \mathbb{R}^b_+ \) such that \( X_i \subset x'_j + \mathbb{R}^b_+ \). If \( x_i \in \hat{X}_i(\lambda) \), there exists \( x_j \in X_j \), for every \( j \neq i \), such that \( \sum_{j \in \mathcal{I}} x_j = \sum_{j \in \mathcal{I}} e_j \). Consequently,

\[
x'_i \leq x_i = -\sum_{j \neq i} x_j + \sum_{j \in \mathcal{I}} e_j \leq -\sum_{j \neq i} x'_j + \sum_{j \in \mathcal{I}} e_j
\]

and thus \( \hat{X}_i(\lambda) \) is bounded.
We now show that $\bar{Z}_i(\lambda)$ is bounded under the assumptions $rg(V) = \#J$ and $R$. Indeed, for every $z_i \in \bar{Z}_i(\lambda)$, there exists $(z_{i,j})_{j \neq i} \in \prod_{j \neq i} Z_j$, $(x_{i,j})_{j} \in \prod_{j \in I} X_j$, $p \in B_L(0, 1)$, $q \in \mathbb{R}^J$ such that $^q W(q) \lambda \in B_{I'}(0, 1)$, $\sum_{j \in I} z_j = 0$ and $(x_{j}, z_j) \in B^{+}_{x_F}(p, q)$ for every $j \in I$. As $(x_{i,j}, z_i) \in B^{+}_{x_F}(p, q)$ and $(x_{i,p}) \in \bar{X}_i(\lambda) \times B_L(0, 1)$, a compact set, there exists $\alpha_i \in \mathbb{R}^B$ such that

$$\alpha_i \leq p \circ (x_i - e_i) \leq W(q) z_i.$$

But, using the fact that $\sum_{i \in I} z_i = 0$, we also have

$$W(q) z_i = W(q) \left( - \sum_{j \neq i} z_j \right) \leq - \sum_{j \neq i} \alpha_j,$$

hence there exists $r > 0$ such that $W(q) z_i \in B_0(0, r)$. Since $rang V = \#J$ and Assumption R holds true, $rang W(q) = \#J$. Lemma 2 applied to $W(q)$ for $q \in A := \{ q \in \mathbb{R}^J : ^q W(q) \lambda \in B_{I'}(0, 1) \}$, which is a compact subset, implies that there exists $c > 0$ such that, for every $q \in A$, $z_i \in \mathbb{R}^J$, $c \| z_i \| \leq \| W(q) z_i \|$ hence

$$c \| z_i \| \leq \| W(q) z_i \| \leq r$$

for every $z_i \in \bar{Z}_i(\lambda)$, which shows that $\bar{Z}_i(\lambda)$ is a bounded set. Since $\bar{X}_i(\lambda)$ and $\bar{Z}_i(\lambda)$ are bounded sets for every $i$, $B(\lambda)$ is also bounded. □

**Proof of Proposition 8** Let $(\bar{x}, \bar{z}', \bar{p}, \bar{q}')$ be an equilibrium of the financial exchange economy $(\Sigma, \mathcal{F}')$. Taking into account the definition of an equilibrium, we will prove that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium financial exchange economy $(\Sigma, \mathcal{F})$ where $\bar{q}$ is the no-arbitrage price of the financial structure $\mathcal{F}$ which is defined with the same $\lambda \in \mathbb{R}^{I'}_{++}$ as $q'$, with $\bar{z} = (\bar{z}_i)_{i \in I}$ such that for $i \in I$, $\bar{z}_i' = \bar{z}_i''$ for $j \in J'$ and $\bar{z}_i' = 0$ for $j \in J \setminus J'$.

- $\forall i \in I$, $(\bar{x}_i, \bar{z}_i) \in B^{+}_{x_F}(\bar{p}, \bar{q})$. Indeed, from the definition of $\bar{z}_i$,

$$[W_{x_F}(q') \bar{z}_i'](\xi) = [W_{x_F}(\bar{q}) \bar{z}_i](\xi),$$

so for all $\xi \in \mathbb{D},$

$$\bar{p}(\xi) \bullet \mathbb{R} [\bar{x}_i(\xi) - e_i(\xi)] \leq [W_{x_F}(q') \bar{z}_i'](\xi) = [W_{x_F}(\bar{q}) \bar{z}_i](\xi).$$

- $\forall i \in I$, $[P^i(\bar{x}) \times Z_i] \cap B^{+}_{x_F}(\bar{p}, \bar{q}) = \emptyset$. Let us argue by contradiction. If there exists $i \in I$ and $(x_i, z_i) \in [P^i(\bar{x}) \times Z_i] \cap B^{+}_{x_F}(\bar{p}, \bar{q})$ this is equivalent to $x_i \in P^i(\bar{x})$ and for all $\xi \in \mathbb{D}$, $\bar{p}(\xi) \bullet \mathbb{R} [x_i(\xi) - e_i(\xi)] \leq [W_{x_F}(\bar{q}) z_i](\xi).$

By definition of $\bar{q}$, for $j \in J'$, $W^{J'}_{x_F}(q') = W^{J'}_{x_F}(\bar{q})$. So, $\langle W_{x_F}(q') \rangle \subset \langle W_{x_F}(\bar{q}) \rangle$. In addition, thanks to Lemma 1, Assumption R is also true on the financial structure $\mathcal{F}'$ thus $rank W_{x_F}(q') = rank V' = \#J'rank V = rank W^{J'}_{x_F}(\bar{q})$. Hence $\langle W_{x_F}(q') \rangle = \langle W_{x_F}(\bar{q}) \rangle$.

So, there exists $z_i' \in Z_i' = \mathbb{R}^J$ such that $W_{x_F}(q') z_i' = W_{x_F}(\bar{q}) z_i$, thus for all $\xi \in \mathbb{D},$

$$[W_{x_F}(\bar{q}) z_i](\xi) = [W_{x_F}(q') z_i'](\xi).$$
and this implies that
\[
\forall i \in \mathcal{I}, \forall \xi \in \mathbb{D}, \bar{p}(\xi) \bullet [\bar{x}_i(\xi) - e_i(\xi)] \leq [W_{\mathcal{F}'}(q') \bar{z}_i'](\xi)
\]
that is \((x_i, z'_i) \in B'_{\mathcal{F}', (\bar{p}, q')}\). Since \((x_i, z'_i) \in [P^i(\bar{x}) \times Z'_i]\), this contradicts the fact that \((\bar{x}, \bar{z}', \bar{p}, q')\) is an equilibrium of the financial exchange economy \((\Sigma, \mathcal{F}')\).

- \(\sum_{i \in \mathcal{I}} \bar{x}_i = \sum_{i \in \mathcal{I}} e'_i\), from the market clearing condition in the economy \((\Sigma, \mathcal{F}')\) and \(\sum_{i \in \mathcal{I}} \bar{z}_i = 0\) from the market clearing condition on the asset markets of \(\sum_{i \in \mathcal{I}} \bar{z}_i = 0\) and the definition of \(\bar{z}_i\). \(\square\)
References


