Optimal Risk Sharing with Optimistic and Pessimistic Decision Makers

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Main motivation: we aim at proving that under mild conditions individually Pareto rational optima will exist even in presence of non-convex preferences. We essentially consider preferences on $\ell^\infty_+$ hence accommodating the behavior of DM in front of a countable flow $x$ of payoffs, or else choosing among financial assets $x$ whose outcomes depend on the realization of a countable set of states of the world. Therefore, our conditions for the existence of P.O. can be interpreted as myopia in the first context and as pessimism or reasonable optimism in the second context.
Replacing the uncertain context by a risky one, we then derive some necessary properties of risk sharing. For strong risk averters these conditions are exactly the usual ones of pairwise comonotonicity, for strong risk lovers exclusively corner conditions or pairwise comonotonicity can appear. A crucial result for our work is the proof that the closed unit ball $\bar{B}(0, 1)$ for the normed dual $\ell^\infty$ of the normed space $\ell^1$ is compact in the Mackey topology $\tau(\ell^\infty, \ell^1)$. 
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Preliminary results

Theorem

Let $\mathcal{B}(S)$ be the normed space of bounded real mappings on $S$ endowed with the sup-norm. Then:

1) $\mathcal{B}(S)$ is the norm dual of the space $\text{rca}(S)$ of all regular and bounded Borel measures for $S$ endowed with the discrete topology.

2) The closed unit ball of $\mathcal{B}(S)$ is compact for the Mackey topology $\tau(\mathcal{B}(S), \text{rca}(S))$.

Corollary

The closed unit ball $\bar{B}(0, 1)$ for the normed dual $\ell^\infty$ of the normed space $\ell^1$ is compact in the Mackey topology $\tau(\ell^\infty, \ell^1)$.

Proof. Taking $S = \mathbb{N}$, clearly $\ell^\infty = \mathcal{B}(S)$. 

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Existence of individually rational Pareto efficient allocations under Mackey upper-semicontinuous preferences

Assumption

(1) For every $i = 1, \ldots, m$, $\succeq_i$ is a transitive reflexive preorder on $\ell_+^\infty$.

(ii) For every $i$ and every $y_i \in \ell_+^\infty$, \[ \{ x_i \in \ell_+^\infty \mid x_i \succeq_i y_i \} \text{ is } \tau\text{-closed where } \tau \text{ is the Mackey topology } \tau(\ell^\infty, \ell^1). \]

Theorem

Under Assumptions (1) and (2), individually rational Pareto efficient allocations exist.
Examples of Mackey upper-semicontinuous preferences

**Definition**

Let \((S, \mathcal{A})\) be a measurable space and \(B_{\infty}(\mathcal{A})\) be the space of bounded \(\mathcal{A}\)-measurable mappings from \(S\) to \(\mathbb{R}\).

\(\nu\) is a capacity on \(\mathcal{A}\) if \(\nu(\emptyset) = 0\), \(\nu(S) = 1\), \(\forall A, B \in \mathcal{A}, A \subset B \Rightarrow \nu(A) \leq \nu(B)\).

\(\forall x \in B_{\infty}(\mathcal{A})\), the Choquet integral of \(x\) with respect to \(\nu\), denoted \(\int x d\nu\) is defined by

\[
\int_{-\infty}^{0} (\nu(x \geq t) - 1) dt + \int_{0}^{+\infty} \nu(x \geq t) dt.
\]
>From Araujo et al. “General Equilibrium, Wariness and efficient bubbles", JET forthcoming, we know that preferences on $\ell^\infty$ represented by a Choquet integral is Mackey usc if and only if $\succeq$ is strongly myopic where:

**Definition**

A preference relation $\succeq$ on $\ell^\infty$, is strongly myopic if

$\forall (x, y) \in \ell^\infty \times \ell^\infty \; x \succeq y, \; z \in \ell^\infty_+$, implies that there exists $n$ large enough such that $x \succeq y + z_{E_n}$ where $z_{E_n}(m) = 0$ if $m < n$ and $z(m)$ if $m \geq n$. 
Pessimistic Mackey upper-semicontinuous preferences

Definition

A Choquet DM will be said pessimistic if $\nu$ is convex, i.e.,

$$\forall A, B \in \mathcal{A}, \nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B).$$

Recall that if $\nu$ is convex, then

$$\int x \, dv = \min \{ E_P(x) \mid P \text{ additive probability measure } \geq \nu \}$$
Proposition

For a pessimistic decision maker, the following assertions are equivalent:

1. \( \succeq \) is Mackey upper-semicontinuous;
2. \( v \) is outer continuous, i.e., \( \forall A_n, A \in \mathcal{A} = \mathcal{P}(\mathbb{N}), A_n \downarrow A \Rightarrow v(A_n) \downarrow v(A) \).

Example \( I(x) = \inf_{\mathbb{N}} \{x(n)\} \) or equally \( I(x) = \int x \, dv \) where \( v(A) = 0 \ \forall A \neq \mathbb{N}, v(\mathbb{N}) = 1 \).

Remark that \( I \) is not Mackey lower semicontinuous.
Optimistic Mackey upper-semicontinuous preferences

Definition

A Choquet DM will be said optimistic if \( v \) is concave, i.e.,
\[
\forall A, B \in \mathcal{A}, \quad v(A \cup B) + v(A \cap B) \leq v(A) + v(B).
\]

Recall that if \( v \) is concave,
\[
\int x dv = \max \{ E_P(x) \mid P \text{ additive probability measure } \geq v \}
\]
or else \( x, y \in \mathcal{B}_\infty(\mathcal{A}), \ x \sim y, \ \alpha \in ]0, 1[ \Rightarrow \alpha x + (1 - \alpha) y \preceq y. \)
Proposition

For an optimistic decision maker the following assertions are equivalent:

(1) $\succeq$ is Mackey upper-semicontinuous;
(2) $\preceq$ is Mackey lower-semicontinuous;
(3) $\nu$ is outer continuous at $\emptyset$, i.e., $A_n \in \mathcal{P}(\mathbb{N})$, $A_n \downarrow \emptyset \Rightarrow \nu(A_n) \downarrow 0$.

Remark. $I(x) = \sup_{N} \{x(n)\}$, i.e., $I(x) = \int x dv$ with $\nu(A) = 1 \ \forall A \neq \emptyset, \nu(\emptyset) = 0$ is not Mackey upper-semicontinuous.
Example where no existence of nontrivial individually rational Pareto optima if such a “too optimistic” DM is present

Consumption space $X = \ell^\infty_+, p_s = \frac{1}{2^s}$.
DM1 $U_1(x) = \sum_{s \in \mathbb{N}} p_s x_s$, initial endowments $\omega^1 = (2 - \frac{1}{s+1})_{s \in \mathbb{N}}$
DM2 $U_2(x) = \sup_{s \in \mathbb{N}} x_s$, $\omega^2 = \omega^1$.

The unique Pareto optimal allocation is $(\bar{x}^1 = \omega^1 + \omega^2, \bar{x}^2 = 0)$. No individually rational Pareto optimal allocation.
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Optimal risk sharing for risk lovers and risk averse decision makers

Interpret $\ell^\infty$ as the set of all bounded $\mathcal{A}$ measurable mappings $x$ from $\mathbb{N}$ to $\mathbb{R}$ where $\mathcal{A} = \mathcal{P}(\mathbb{N})$ and assume that a $\sigma$-additive probability $P$ is given on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. So any $x$ can now be interpreted as a random variable.

**Definition**

$x$ is less risky than $y$ for the second order stochastic dominance denoted $x \preceq_{SSD} y$ if

$$\int_{-\infty}^{t} P(x \leq u) du \leq \int_{-\infty}^{t} P(y \leq u) du$$

for all $t \in \mathbb{R}$.

$x$ is strictly less risky than $y$ denoted $x \succ_{SSD} y$ if one of the previous inequality is strict.
Definition

Let $\succeq_i$ be the preference relation of a DM.

$\succeq_i$ is a strict (strong) risk averter if

$x \succeq_{SSD} y \Rightarrow x \succeq_i y$ and $x \succ_{SSD} y \Rightarrow x \succ_i y$.

$\succeq_i$ is a strict (strong) risk lover if

$x \succeq_{SSD} y \Rightarrow x \preceq_i y$ and $x \succ_{SSD} y \Rightarrow x \prec_i y$. 
Examples of Mackey upper-semicontinuous strict risk lover of strict risk averter

Consider a Choquet decision maker where $\nu = f \circ P$, i.e., $\nu$ is a distortion of a probability $P$, i.e., $f : [0, 1] \rightarrow [0, 1]$, $f(0) = 0$, $f(1) = 1$, $f$ strictly increasing.

If $f$ is strictly concave and continuous, then:
$x \in \ell^\infty \rightarrow I(x) = \int x d\nu_f$ with $\nu_f = f \circ P$ defines a Mackey upper-semicontinuous strict risk lover.

If $f$ is strictly convex and continuous, then:
$x \in \ell^\infty \rightarrow I(x) = \int x d\nu_f$ with $\nu_f = f \circ P$ defines a Mackey upper-semicontinuous strict risk averter.
The main result

Theorem

Consider $m$ strict risk averters $i = 1, \ldots, m$ with Mackey upper-semicontinuous preferences $\succeq_i$ and $n$ strict risk lovers $j = 1, \ldots, n$ with Mackey upper-semicontinuous preferences $\succeq_j$ with initial endowments $\omega_i \in \ell^{\infty}_{++}, \omega_j \in \ell^{\infty}_{++}$. Then individual rational Pareto efficient allocations (in $(\ell^{\infty}_{++})^{m+n}$) exist.
Theorem continued

**Theorem**

For such PO \((x_i, i = 1, \ldots, m; y_j, j = 1, \ldots, n)\) we have:

1) The allocations of risk averters are pairwise comonotonic, i.e.,
\[
(x_{i_1}(s) - x_{i_1}(t))(x_{i_2}(s) - x_{i_2}(t)) \geq 0 \quad \forall (s, t) \in \mathbb{N}^2, \\
\forall (i_1, i_2) \in \{1, \ldots, m\}^2.
\]

2) For two risk lovers \(j_1, j_2:\)
\[
(y_{j_1}(s) - y_{j_1}(t))(y_{j_2}(s) - y_{j_2}(t)) < 0 \Rightarrow \\
(y_{j_1}(s) \wedge y_{j_1}(t))(y_{j_2}(s) \wedge y_{j_2}(t)) = 0 \quad \text{(i.e. in at least one state, one of these two risk lovers gets 0).}
\]

3) If all allocations of risk lovers are strictly positive, then these allocations are pairwise comonotonic.