Nonlinear Dynamics and Bifurcation

Macroeconometric and Financial Models
Finding of Chaos in Economic Data

The Controversy

(The 18 papers on next slide is a small subset)


The Solution

• “A Single-Blind Controlled Competition among Tests for Nonlinearity and Chaos”

by W. A Barnett, A. R. Gallant, M. Hinich, J. A. Jungeilges, D. Kaplan, and M. Jensen

*Journal of Econometrics*, vol. 82, 1997, pp. 157-192
Models Used to Produce Simulated Data

- **Model 1**: Chaotic Feigenbaum recursion
- **Model 2**: GARCH process
- **Model 3**: Nonlinear moving average (NLMA)
- **Model 4**: ARCH process
- **Model 5**: Linear ARMA process
**Model 1** is the fully deterministic, chaotic Feigenbaum recursion of the form:

\[ y_t = 3.57y_{t-1}(1 - y_{t-1}) \]

where the initial condition was set at \( y_0 = 0.7 \).

**Model 2** is a GARCH process of the following form:

\[ y_t = h_t^{1/2}u_t \]

where \( h_t \) is defined by

\[ h_t = 1 + 0.1y_{t-1}^2 + 0.8h_{t-1} \]

with \( h_0 = 1 \) and \( y_0 = 0 \).
**Model 3** is the nonlinear moving average (NLMA):

\[ y_t = u_t + 0.8 u_{t-1} u_{t-2} \]

**Model 4** is an ARCH process of the following form:

\[ y_t = (1 + 0.5 y_{t-1}^2)^{\frac{1}{2}} u_t \]

with the initial observation set at \( y_0 = 0 \).

**Model 5** is an ARMA linear model of the form:

\[ y_t = 0.8 y_{t-1} + 0.15 y_{t-2} + u_t + 0.3 u_{t-1} \]

with \( y_0 = 1 \) and \( y_1 = 0.7 \).
Tests Included in the Competition

- Melvin Hinich’s bispectrum test
- The BDS (Brock, Dechert, Scheinkman, and LeBaron) correlation function test
- Hal White’s neural net test
- Danny Kaplan’s surrogate data test
- The NEGM Liapunov exponent test
Let \( \{c_{xxx}(r,s)\} \) be the third order moments in the time domain. For frequencies, \( f_1 \) and \( f_2 \), the bispectrum, \( B_{xxx}(f_1,f_2) \), is defined by

\[
B_{xxx}(f_1,f_2) = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} c_{xxx}(r,s) \exp[-i2\pi(f_1r + f_2s)]
\]
Let $S_{xx}(f)$ be the ordinary power spectrum of $x(t)$ at frequency $f$. Then the skewness function is the normalized bispectrum:

$$\Gamma^2(f_1,f_2) = \left| B_{xxx}(f_1,f_2) \right|^2 / S_{xx}(f_1)S_{xx}(f_2)S_{xx}(f_1+f_2)$$
For these tests, the relevant class of models is the general univariate nonlinear causal time series models. The process, $X(t)$, then can be viewed as the “output” of a nonlinear operator, $F$, on an “input” stationary process, $\{\varepsilon(t)\}$:

$$X(t) = F[\varepsilon(t), \varepsilon(t-1), \ldots].$$
Volterra Expansion:
with Volterra kernals, $a(i,j,k)$
Norbert Wiener (1985)

$$X(t) = \sum_{i=0}^{\infty} a(i) \varepsilon(t-i) + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a(i,j) \varepsilon(t-i) \varepsilon(t-j)$$

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a(i,j) \varepsilon(t-i) \varepsilon(t-j) \varepsilon(t-k) + ...$$
Definitions of Whiteness and Nonlinearity

- **White Noise:**
  If \( \{x(t)\} \) is a zero mean third order stationary time series, then the mean, \( \mu_x = E[x(t)] = 0 \), the second order autocovariance, \( c_{xx}(m) = E[x(t+m)s(t)] \), and the third order autocovariances, \( c_{xxx}(s,r) = E[x(t+r)s(t+s)x(t)] \) are independent of \( t \). If \( c_{xx}(m) = 0 \) for all nonzero \( m \), the series is white news.

- **Pure White Noise:**
  We define a pure (also called “strict” sense) white noise series as a white noise process in which \( (x(n_1), \ldots, x(n_N)) \) are independent random variables for all values of \( n_1, \ldots, n_N \).
**Linear Process:**
A linear stochastic process is a linear filter of i.i.d inputs.

[An ARIMA process is a finite dimensional linear filter, while a first order Volterra expansion is infinite dimensional and spans the space of linear filters. In some definitions of linearity, the innovations are assumed to be white noise martingale differences, rather than are i.i.d. inputs]

**Linearity in the Mean:**
A process is “linear in the mean” relative to an information set, if the process has a conditional mean function that is a linear function of the elements of the information set.

[The information set usually contains lagged observations on the process. A process that is not linear in the mean is said to exhibit “neglected nonlinearity.”]
Lack of Third Order Nonlinear Dependence:

A process exhibits third order nonlinear dependence, if the skewness function in the frequency domain is not flat as a function of frequency pairs.

[This form of nonlinearity is called third order, since the skewness function is a normalization of the Fourier transform of the third order autocovariances. That Fourier transformation is called the bispectrum, and is the third order polyspectrum].
Paragraph 8 of the Uploaded Papers section at:

http://econ.tepper.cmu.edu/barnett/Papers.html
Papers with Yijun He on the UK Continuous Time Macroeconometric Model (Bergstrom et al)


Model structure:
14 second order differential equations

Structural parameters:
63 structural parameters, including 27 long-run propensities and elasticities and 33 speed of adjustment parameters

Free parameters:
3 trend parameters
Hopf Bifurcation Example

- \( \mathbf{D}_x = -y + x(\vartheta - (x^2 + y^2)) \)
- \( \mathbf{D}_y = x + y(\vartheta - (x^2 + y^2)) \)

The equilibria are found by setting \( \mathbf{D}_x = \mathbf{D}_y = 0 \).

All equilibria satisfy \( x^* = y^* = 0 \), with the stable equilibria occurring for \( \vartheta < 0 \) and the unstable equilibria occurring for \( \vartheta > 0 \).
Phase Diagram for Hopf Bifurcation
Transcritical Bifurcation Example

• \( \mathbf{Dx} = \vartheta x - x^2 \)

- It is stable around the equilibrium \( x^* = 0 \) for \( \vartheta < 0 \), and unstable for \( \vartheta > 0 \). The equilibrium \( x^* = \vartheta \) is stable for \( \vartheta < 0 \), and unstable for \( \vartheta > 0 \).
The solid dark lines represent stable equilibria, while the dashed lines display unstable ones.

• **Transcritical bifurcation diagram:**
Bergstrom UK Model: 2-dimensional sections of stable region

- $\Theta_1$ is the confidence region.
- $\Theta$ is the theoretically feasible region.
$\theta_6$ versus $\theta_2$

(a) Bifurcation within $\Theta_1$  
(b) Bifurcation within $\Theta$
$\theta_{23}$ versus $\theta_{62}$

(a) Bifurcation within $\Theta_1$

(b) Bifurcation within $\Theta$

transcritical bifurcation

stable region

Hopf bifurcation

stable region
• $\Theta_1$ is the confidence region.

• $\Theta$ is the theoretically feasible region.
$\theta_2, \theta_{23}, \theta_{62}$

(a) Bifurcation within $\Theta_1$

(b) Bifurcation within $\Theta$
$\theta_{12}$, $\theta_{23}$, $\theta_{32}$

(a) Bifurcation within $\Theta_1$

(b) Bifurcation within $\Theta$

transcritical bifurcation

Hopf bifurcation
Bergstrom’s Fiscal Policy Design

- **Instrument:** total taxation variable
- **Target:** real net output
- **Policy intent:** Seeks to use the instrument to stabilize the target with the ultimate objective of stabilizing the economy's dynamics. The stabilization policy rule is closed loop, feeding back observed values of the target.
Symbols in Figures

**Fiscal Policy Rule Parameters:**

- $\beta$ = strength of feedback
- $\gamma$ = speed of policy adjustment

The adjustment lag is caused by delays in sampling the target variable and delays in adjusting the instrument to the observed target variable.
Private Sector Parameters in Figures

- $\theta_2 = \text{coefficient of the real interest rate in the consumption function.}$

- $\theta_5 = \text{a measure of the importance of capital in production.}$

- $\theta_{62} = \text{rate of growth of expected labor supply trend.}$
Parameter Settings

• Private sector parameters set at their estimated values.

• Parameters of fiscal policy feedback rule set at various settings.
$\theta_{62}$ versus $\theta_2$
Leeper and Sims Euler Equations Model of the US Economy

- Deep parameters solve Lucas critique.
- Model first appeared in:

Our Findings with the Leeper and Sims Model

The Leeper and Sims model consists of differential equations with algebraic constraints. We find the existence of a singularity bifurcation boundary near the model’s parameter point estimates. To our knowledge, this kind of bifurcation has not previously been observed in macroeconomics.
Singularity Bifurcation

- As parameters approach a singularity boundary, one eigenvalue of the linearized part of the model rapidly moves to infinity, while others remain bounded. This implies nearly instantaneous response of some variables to changes of other variables.

- On the singularity boundary, the number of differential equations will decrease, while the number of algebraic constraints will increase. Singularity bifurcations thereby cause a change in the order of the dynamics.
Consider a continuous time model in the following form:

\[ A(x(t), \theta)Dx(t) = F(x(t), \theta), \]

in which \( x(t) \) is the state vector, \( D \) is the differentiation operator, \( t \) is time, and \( A(x(t), \theta) \) is a matrix valued function of time. The most noteworthy aspect of the form is the possibility that the matrix \( A \) be singular.
Singularity-induced bifurcation occurs when the rank of $A(x(t), \theta)$ change, such as from an invertible matrix to a singular one. In such cases, the dimension of the dynamical part of the system changes.
Example of Singularity Bifurcation

- \( D_x = ax - x^2, \)
- \( \vartheta D_y = x - y^2, \)

In which \( a > 0 \). The equations consist of two differential equations with no algebraic equations for nonzero \( q \). But when \( \vartheta = 0 \), the system has one differential equation and one algebraic equation.
By setting $Dx = Dy = 0$, we can find that for every $\vartheta$, the equilibria are at $(x,y) = (0,0)$ and $(x,y) = (a,\pm a^{1/2})$. In this case, the system is unstable around the equilibrium $(x^*,y^*) = (0,0)$ for any value of $\vartheta$. The equilibrium $(x^*,y^*) = (a,+a^{1/2})$ is unstable for $\vartheta < 0$ and stable for $\vartheta > 0$. The third equilibrium $(x^*,y^*) = (a,-a^{1/2})$ is unstable for $\vartheta > 0$ and stable for $\vartheta < 0$. 
The figure below is with a normalization at $a = 1$ with positive. When $\theta$ is negative, the figure remains valid, but with the diagram flipped over about the x axis, so that $(1,1)$ becomes unstable and $(1,-1)$ becomes stable.

The equilibrium $(0,0)$ remains unstable for either positive or negative $\theta$. 
Singularity Bifurcation Phase Portrait with $\theta > 0$
• Recall that $Dx = ax - x^2$ and $\vartheta Dy = x - y^2$.

• When $\vartheta = 0$, the system behavior degenerates into movement along the one dimensional curve $x - y^2 = 0$, as shown in the figure below, with $(0,0)$ being an unstable equilibrium and both $(1,1)$ and $(1,-1)$ being stable equilibria. Note that the second equation changes from a differential equation to an algebraic equation.
Phase Portrait with $\theta = 0$ on Singularity Bifurcation Boundary
Bifurcation of New Keynesian Models

- Research joint with Evgeniya A. Duzhak.
- Three economic agents:
  - Households
  - Firms
  - Central Banks
- Linearize around zero inflation steady state.
Linearized Model

Three Equations:

- Phillips curve relating inflation to output gap. The output gap is the gap between the actual sticky prices output and the flexible-price equilibrium output.
- An IS (investment-savings) curve determining the output gap.
- A monetary policy rule
Monetary Policy Rules

- **Taylor rules:**
  - Feed back inflation rate and output gap to set interest rate

- **Inflation targeting:**
  - Feed back only the inflation rate as a final target, in setting the interest rate.
Taylor Rule Types

- **Current looking**: \( i_t = a_1 \pi_t + a_2 x_t \)
- **Backward looking**: \( i_t = a_1 \pi_{t-1} + a_2 x_{t-1} \)
- **Forward looking**: \( i_t = a_1 \pi_{t+1} + a_2 x_{t+1} \)
- **Hybrid**: \( i_t = a_1 \pi_{t+1} + a_2 x_t \)

➢ where \( i_t \) = interest rate, \( \pi_t \) = inflation rate, and \( x_t \) = output gap.
Taylor Rules with Inflation Rate Smoothing

- **Current looking:**
  \[ i_t = a_1 \pi_t + a_2 x_t + a_3 i_{t-1} \]

- **Backward looking:**
  \[ i_t = a_1 \pi_{t-1} + a_2 x_{t-1} + a_3 i_{t-1} \]

- **Forward looking:**
  \[ i_t = a_1 \pi_{t+1} + a_2 x_{t+1} + a_3 i_{t-1} \]

- **Hybrid:**
  \[ i_t = a_1 \pi_{t+1} + a_2 x_t + a_3 i_{t-1} \]
Inflation Targeting Types

• **Current looking:** \( i_t = a_1 \pi_t \)
• **Backward looking:** \( i_t = a_1 \pi_{t-1} \)
• **Forward looking:** \( i_t = a_1 \pi_{t+1} \)
• A Taylor rule or an inflation targeting rule is called “active,” if the coefficient of the inflation rate, $a_1$, exceeds one.

• A Taylor rule or an inflation targeting rule is called “passive,” if the coefficient of the inflation rate, $a_1$, is less than one.
Flip Bifurcation

• Also called period doubling bifurcation.
• The number of frequencies in the power spectrum doubles, when a flip bifurcation boundary is crossed.
• Possible only in discrete time.
• Made famous by the Feigenbaum recursion.
## Results with New Keynesian Models: Types of Bifurcation Found with Each

<table>
<thead>
<tr>
<th>Version of Rule</th>
<th>Taylor Rule</th>
<th>Taylor Rule with Interest Smoothing</th>
<th>Inflation Targeting</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current looking</td>
<td>Hopf bifurcation</td>
<td>Hopf bifurcation Flip bifurcation</td>
<td>Hopf bifurcation</td>
</tr>
<tr>
<td>Backward looking</td>
<td>Hopf bifurcation</td>
<td>Hopf bifurcation Flip bifurcation</td>
<td>Hopf bifurcation</td>
</tr>
<tr>
<td>Forward looking</td>
<td>Hopf bifurcation Flip bifurcation</td>
<td>No bifurcation boundaries found within theoretically feasible region.</td>
<td>Hopf bifurcation</td>
</tr>
<tr>
<td>Hybrid rule</td>
<td>Hopf bifurcation</td>
<td>Hopf bifurcation Flip bifurcation</td>
<td>Does not apply</td>
</tr>
</tbody>
</table>
INSIDE THE ECONOMIST’S MIND
Conversations with Eminent Economists
Edited by Paul A. Samuelson and William A. Barnett

“Scholars will value these interviews as primary sources. Readers with only a passing interest in economics will be delighted by their entertaining insights into the minds and lives of these great thinkers. This is one of the most valuable projects in academic economic publishing.”

- Professor Douglas Gale, NYU

November 2006 • 6 x 9 in • 400 pages • 75 photos
1-4051-5917-0 / 978-1-4051-5917-3 • PB • $29.95
1-4051-5715-1 / 978-1-4051-5715-5 • HB • $74.95

Blackwell Publishing
1-800-216-2522
WWW.BLACKWELLPUBLISHING.COM
A lively collection of interviews with premier economists

“In candid interviews, these great economists prove to be fabulous story tellers of their lives and times. Unendingly gripping for insiders, this book should also help non-specialists understand how economists think.”

Julio Rotemberg, Harvard University Business School

Published: November 2006 / 456 pages
978-1-4051-5917-3 PB £19.99
978-1-4051-5715-5 HB £60.00

Order your copy at www.blackwellpublishing.com
The common thread in Samuelson and Barnett’s, *Inside the Economist’s Mind*:
Jacques Drèze
Tom Sargent