Existence of Equilibria in Financial Markets With Restricted Participation

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(Job Market Paper)

Abstract
Investors facing restrictions on the portfolios that they can trade, is more of a norm than an exception. We consider a model in which investors’ portfolio sets are constrained. As in Balasko, Cass and Siconolfi (1990) these constraints are exogenously given (possibly arising due to some institutional reasons). Moreover, we consider very general restrictions on portfolio sets as in Siconolfi (1986), where each agent’s portfolios set is assumed to be convex and containing zero.

In two date (one period) models without restrictions on portfolio sets, the existence issue has been extensively studied. Cass (1984) and Werner (1985, 1989) showed existence with nominal assets. Duffie and Shafer (1985) showed a generic existence result with real assets. This second approach has been extensively used as surveyed in Magill and Shafer [14].

This paper primarily examines the characterization of equilibrium asset prices with arbitrage free asset prices, in a multiperiod model when investors face such general portfolio restrictions. In the absence of such portfolio constraints the approach initiated by Cass (1984), has been extensively used to characterize equilibrium asset prices with arbitrage free asset prices. See Cass (1984), Duffie (1987) and Florenzano and Gourdel (1994). Moreover this approach is also useful in showing the existence of an equilibrium. See Magill and Shafer (1991), Florenzano and Gourdel (1994), Magill and Quinzii (1996) and Angeloni and Cornet (2006) among others. Another approach to prove existence in a differentiable economy is to show existence in a numeraire asset economy and infer the existence in the nominal asset economy (See Villanacci et al. 2002 and Magill and Quinzii 1996).

In the approach of Cass (1984), one agent is assumed to be unrestricted and hence behaves as in an Arrow-Debreu world. This assumption breaks the symmetry of the problem and thus the proof is not symmetric with respect to the agents. Hahn and Won (2003) are able to avoid this Cass approach, albeit with monotonic preferences and a more involved notion of ‘projective’ arbitrage. In a recent working paper, Da-Rocha and Triki (2005) have been able to show this characterization in a symmetric manner. We follow this approach, but with a more general notion of absence of market arbitrage and a more general compatibility condition.

Keywords: Exchange Economies, incomplete markets, financial equilibria, constrained portfolios, multiperiod models, arbitrage free asset prices.

JEL Classification: C62, D52, D53, G11, G12.

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1 Introduction

Investors facing restrictions on the portfolios that they can trade, is more of a norm than an exception. We consider a model in which investors’ portfolio sets are constrained. As in Balasko, Cass and Siconolfi [2] these constraints are exogenously given (probably arising due to some institutional reasons). Moreover, we consider very general restrictions on portfolio sets as in Siconolfi [22], where each agents portfolios set is assumed to be convex and containing zero.

This paper primarily examines the existence of a financial equilibrium in a multiperiod model when investors face such general portfolio restrictions. In two date (one period) models without restrictions on portfolio sets, the existence issue has been extensively studied. Cass ([4]) and Werner ([24], [25]) showed existence with nominal assets. Duffie and Shafer ([9]) showed a generic existence result with real assets. This second approach has been extensively used. Magill and Shafer [14] provide a good survey of financial markets equilibria and contingent markets equilibria. Another approach to prove existence in a differentiable economy is to show existence in a numeraire asset economy and infer the existence in the nominal asset economy (See Villanacci et al. Villanacci et al and Magill and Quinzii [15]).

Multiperiod models are better equipped to capture the evolution of time and uncertainty and are a necessary step before studying infinite horizon models. Following Debreu’s [7] pioneering model we consider an event-tree to represent the evolution of time and uncertainty. Magill and Quinzii ([15]) and Angeloni and Cornet ([1]) are great references for the treatment of multiperiod financial models. Each node in the event tree represents a date event. Given information on asset prices and spot prices at all date events, consumers will choose a consumption and a portfolio of assets (assumed to be constrained here), such that the node specific value of the consumption does not exceed the node specific value of their endowments and the net returns from the portfolio.

In the absence of such portfolio restrictions the notion of absence of arbitrage is clear - if there is no portfolio that yields nonnegative net returns in all nodes and strictly positive returns in some node. However in the case where all agents face restrictions in their asset market participation, the notion of arbitrage and its absence at the individual level may differ from that at the aggregate level. Angeloni and Cornet (2006)[1] make this distinction. Given asset prices, an agent does not have arbitrage opportunities if she cannot find a portfolio within her constrained portfolio set that yields nonnegative net returns in all nodes and strictly positive returns in some node. On the other hand, there are no arbitrage opportunities in the aggregate, if there is no portfolio in the set of pooled
portfolio sets of all agents that yields nonnegative net returns in all nodes and strictly positive returns in some node.

At an equilibrium there must be no arbitrage at the individual level. A natural question then is will any asset price at which there is no arbitrage be an equilibrium asset price. The objective of this paper is to explore this characterization under general portfolio constraints.

In the absence of portfolio constraints, Cass ([4]), Duffie ([8]) and Florenziano and Gourdel ([10]) show this characterization of equilibrium and arbitrage free asset prices. In the presence of such constraints, the approach initiated by Cass ([4]), where one agent has an unconstrained portfolio set, facilitates the existence proof. This approach has been extensively used to show existence ever since, Magill and Shafer ([14]), Florenziano and Gourdel ([10]), Magill and Quinzii ([15]), Angeloni and Cornet ([1]) among others.

This approach of Cass ([4]), breaks the symmetry of the problem and hence the it is not possible to give a symmetric proof of existence (symmetric with respect to the agents’ problem). More recently in a working paper, with such general portfolio restrictions, Da-Rocha and Triki ([6]) have been able to show the characterization between equilibrium and arbitrage free asset prices without the use of the Cass approach. Hahn and Won ([18]), are also able to show this albeit with monotonic preferences and more involved notion of “Projective” arbitrage. However the notion of arbitrage and its absence in these two papers differ from each other and from the notion considered in this paper, that of Angeloni and Cornet ([1]).

In this paper we explore this characterization issue by showing that any market arbitrage free asset price can be supported as an equilibrium asset price. The approach here is similar to that in Da-Rocha and Triki ([6]), however the notion of absence of arbitrage and the compatibility condition required on the agents portfolio sets are weaker that those in Da-Rocha and Triki ([6]). The compatibility condition we require in this paper is, for any payoff that can be obtained through the pooled portfolio set there is some agent who can a fraction of this payoff through her portfolio set and there is some agent who can obtain a negative of a fraction of this payoff through her portfolio set. We can interpret this as, for any payoff that is possible for all agents pooled together, a fraction of that can be short sold of bought by some agent.

In the Cass approach, the unconstrained agent behaves like in an Arrow-Debreu economy and is able to accommodate the equilibrium excess demand for assets. The attainable consumption and asset allocation are then bounded. We observe that the requirement for attainable asset allocations to be bounded may be unnecessary. If the set of attainable income transfers is bounded then we can guarantee the existence of a weak equilibrium,
which differs from an equilibrium only in the requirement that instead of asset market clearing, there is accounts clearing in the asset markets. The notion of weak equilibrium is thus useful when redundant assets exist.

Section 2 describes the $T$-period model and the notion of a financial equilibrium. Section 3, states the main result and discusses the notion of a weak equilibrium and its existence. Section 3.3 discusses the various notions of absence of arbitrage and the compatibility conditions needed to guarantee the existence of an equilibrium. The previous results in this area are listed as corollaries. Section 4 gives a detailed proof of the central result in this paper.

2 The $T$-period Financial Exchange Economy

2.1 Time and Uncertainty in a Multiperiod Model

We consider a multiperiod exchange economy with $(T + 1)$ dates, $t \in T := \{0, \ldots, T\}$, and a finite set of agents $I = \{1, \ldots, I\}$. The stochastic structure of the model is described by a finite event-tree $D$ of length $T$ and we shall essentially use the same model as Angeloni and Cornet [1], (we refer to [15] for an equivalent presentation with information partitions). The set $D_t$ denotes the nodes (also called date-events) that could occur at date $t$ and the family $(D_t)_{t \in T}$ defines a partition of the set $D$; for each $\xi \in D$ we denote by $t(\xi)$ the unique $t \in T$ such that $\xi \in D_t$. Also we denote the cardinality of the set $D$ by $D$.

At each date $t \neq T$, there is an a priori uncertainty about which node will prevail in the next date. There is a unique non-stochastic event occurring at date $t = 0$, which is denoted $\xi_0$, (or simply 0) so $D_0 = \{\xi_0\}$. Finally, every $\xi \neq \xi_0$ in the event-tree $D$ has a unique predecessor denoted $pr(\xi)$ in $D$. The predecessor mapping $pr : D \setminus \{\xi_0\} \rightarrow D$ satisfies $pr(D_t) = D_{t-1}$, for every $t \neq 0$. The element $pr(\xi)$ is called the immediate predecessor of $\xi$ and is also denoted $\xi^-$. For each $\xi \in D$, we let $\xi^+ = \{\tilde{\xi} \in D : \xi = \tilde{\xi}^-\}$ be the set of immediate successors of $\xi$; we notice that the set $\xi^+$ is nonempty if and only if $\xi \in D \setminus D_T$.

In this paper, we shall use the following notations. A $(D \times I)$-matrix $A$ is an element of $\mathbb{R}^{D \times I}$, with entries $(a(\xi, j))_{\xi \in D, j \in I}$; we denote by $A(\xi) \in \mathbb{R}^I$ the $\xi$-th row of $A$ and by $A(j) \in \mathbb{R}^D$ the $j$-th column of $A$. We recall that the transpose of $A$ is the unique $(I \times D)$-matrix $A^T$ satisfying $(Ax) \cdot y = x \cdot (A^T y)$, for every $x \in \mathbb{R}^I, y \in \mathbb{R}^D$, where $\cdot$ [resp. $\cdot_j$] denotes the usual scalar product in $\mathbb{R}^I$ [resp. $\mathbb{R}^I$]. We shall denote by $\text{rank}_A$ the rank of the matrix $A$. For every subsets $\tilde{D} \subset D$ and $\tilde{I} \subset I$, the $(\tilde{D} \times \tilde{I})$-sub-matrix of $A$ is the $(\tilde{D} \times \tilde{I})$-matrix $\tilde{A}$ with entries $\tilde{a}(\xi, j) = a(\xi, j)$ for every $(\xi, j) \in \tilde{D} \times \tilde{I}$. Let $x, y$ be in $\mathbb{R}^n$; we shall use the notation $x \succeq y$ (resp. $x \succ y$) if $x_h \geq y_h$ (resp. $x_h > y_h$) for every $h = 1, \ldots, n$ and we let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \succeq 0\}, \mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x \succ 0\}$. We shall also use the notation $x \succ y$ if $x \succeq y$ and $x \neq y$. We shall denote by $\| \cdot \|$ the Euclidean norm in the different Euclidean spaces used in this paper and the closed ball centered at $x \in \mathbb{R}^I$ of radius $r > 0$ is denoted $B_L(x, r) := \{y \in \mathbb{R}^I : \|y - x\| \leq r\}$.
Moreover, for $\tau \in T \setminus \{0\}$ and $\xi \in \mathcal{D} \setminus \bigcup_{t=0}^{T-1} \mathcal{D}_t$ we define, by induction, $pr^\tau(\xi) = pr(pr^{\tau-1}(\xi))$ and we let the set of (not necessarily immediate) successors and the set of predecessors of $\xi$ be respectively defined by

$$\mathcal{D}^+(\xi) = \{\xi' \in \mathcal{D} : \exists \tau \in T \setminus \{0\} | \xi = pr^\tau(\xi')\},$$
$$\mathcal{D}^-(\xi) = \{\xi' \in \mathcal{D} : \exists \tau \in T \setminus \{0\} | \xi' = pr^\tau(\xi)\}.$$

If $\xi' \in \mathcal{D}^+(\xi)$ [resp. $\xi' \in \mathcal{D}^+(\xi) \cup \{\xi\}$], we shall also use the notation $\xi' > \xi$ [resp. $\xi' \geq \xi$]. We notice that $\mathcal{D}^+(\xi)$ is nonempty if and only if $\xi \not\in \mathcal{D}_T$ and $\mathcal{D}^-(\xi)$ is nonempty if and only if $\xi \neq \xi_0$. Moreover, one has $\xi' \in \mathcal{D}^+(\xi)$ if and only if $\xi \in \mathcal{D}^-(\xi')$ and similarly $\xi' \in \mathcal{E}^+ \text{ if and only if } \xi \in (\mathcal{E}')^-$. 

2.2 The Stochastic Exchange Economy

At each node $\xi \in \mathcal{D}$, there is a spot market where a finite set $\mathcal{H} = \{1, ..., \mathcal{H}\}$ of divisible physical goods is available. We assume that each good does not last for more than one period. In this model, a commodity is a couple $(h, \xi)$ of a physical good $h \in \mathcal{H}$ and a node $\xi \in \mathcal{D}$ at which it will be available, so the commodity space is $\mathbb{R}^L$, where $L = \mathcal{H} \times \mathcal{D}$. An element $x$ in $\mathbb{R}^L$ is called a consumption, that is $x = (x(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}^L$, where $x(h, \xi) = (x(h, \xi))_{h \in \mathcal{H}} \in \mathbb{R}^\mathcal{H}$, for every $\xi \in \mathcal{D}$.

We denote by $p = (p(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}^L$ the vector of spot prices and $p(\xi) = (p(h, \xi))_{h \in \mathcal{H}} \in \mathbb{R}^\mathcal{H}$ is called the spot price at node $\xi$. The spot price $p(h, \xi)$ is the price paid, at date $t(\xi)$, for the delivery of one unit of the physical good $h$ at node $\xi$. Thus the value of the consumption $x(\xi)$ at node $\xi \in \mathcal{D}$ (evaluated in unit of account of node $\xi$) is

$$p(\xi) \bullet_{\mathbb{R}} x(\xi) = \sum_{h \in \mathcal{H}} p(h, \xi) x(h, \xi).$$

Each agent $i \in \mathcal{I}$ is endowed with a consumption set $X^i \subset \mathbb{R}^L$ which is the set of her possible consumptions. An allocation is an element $x \in \prod_{i \in \mathcal{I}} X^i$, and we denote by $x^i$ the consumption of agent $i$, that is the projection of $x$ onto $X^i$.

The tastes of each consumer $i \in \mathcal{I}$ are represented by a strict preference correspondence $P^i : \prod_{j \in \mathcal{I}} X^j \longrightarrow X^i$, where $P^i(x)$ defines the set of consumptions that are strictly preferred by $i$ to $x^i$, that is, given the consumptions $x^j$ for the other consumers $j \neq i$. Thus $P^i$ represents the tastes of consumer $i$ but also her behavior under time and uncertainty, in particular her impatience and her attitude towards risk. If consumers’ preferences are represented by utility functions $u^i : X^i \longrightarrow \mathbb{R}$, for every $i \in \mathcal{I}$, the strict preference correspondence is defined by $P^i(x) = \{x^i \in X^i | u^i(x^i) > u^i(x')\}$. 

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Finally, at each node $\xi \in \mathcal{D}$, every consumer $i \in I$ has a node-endowment $e^i(\xi) \in \mathbb{R}^H$ (contingent to the fact that $\xi$ prevails) and we denote by $e^i = (e^i(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}^L$ her endowment vector across the different nodes. The exchange economy $\mathcal{E}$ can thus be summarized by

$$\mathcal{E} = [\mathcal{D}; \mathcal{H}; I; (X^i, P^i, e^i)_{i \in I}].$$

### 2.3 The Financial Structure

We consider finitely many financial assets and we denote by $\mathcal{J} = \{1, ..., J\}$ the set of assets. An asset $j \in \mathcal{J}$ is a contract, which is issued at a given and unique node in $\mathcal{D}$, denoted by $\xi(j)$ and called the emission node of $j$. Each asset $j$ is bought (or sold) at its emission node $\xi(j)$ and only yields payoffs at the successor nodes $\xi'$ of $\xi(j)$, that is, for $\xi' > \xi(j)$. We denote by $v(\xi, j)$ the payoff of asset $j$ at node $\xi$. Since we consider only nominal assets this payoff does not depend on the spot prices. For the sake of convenient notations, we shall in fact consider the payoff of asset $j$ at every node $\xi \in \mathcal{D}$ and assume that it is zero if $\xi$ is not a successor of the emission node $\xi(j)$. Formally, we assume that $v(\xi, j) = 0$ if $\xi \not\in \mathcal{D}^+(\xi(j))$. With the above convention, we notice that every asset has a zero payoff at the initial node, that is $v(\xi_0, j) = 0$ for every $j \in \mathcal{J}$. Furthermore, every asset $j$ which is emitted at the terminal date $T$ has a zero payoff, that is, if $\xi(j) \in \mathcal{D}_T$, $v(\xi, j) = 0$ for every $\xi \in \mathcal{D}$.

For every consumer $i \in I$, if $z^i_j > 0$ [resp. $z^i_j < 0$], then $|z^i_j|$ will denote the quantity of asset $j \in \mathcal{J}$ bought [resp. sold] by agent $i$ at the emission node $\xi(j)$. The vector $z^i = (z^i_j)_{j \in \mathcal{J}} \in \mathbb{R}^J$ is called the portfolio of agent $i$.

We assume that each consumer $i \in I$ is endowed with a portfolio set $Z^i \subset \mathbb{R}^J$, which represents the set of portfolios that are admissible for agent $i$.

The price of asset $j$ is denoted by $q_j$ and we recall that it is paid at its emission node $\xi(j)$. We let $q = (q_j)_{j \in \mathcal{J}} \in \mathbb{R}^J$ be the asset price (vector).

**Definition 2.1** A financial asset structure $\mathcal{F} = (\mathcal{J}, (\xi(j), V^j)_{j \in \mathcal{J}}, (Z^i)_{i \in I})$ consists of

- a set of assets $\mathcal{J}$,
- each asset $j \in \mathcal{J}$ is defined by a node of issue $\xi(j) \in \mathcal{D}$ and the vector of returns across all nodes $V^j \in \mathbb{R}^D$,
- a collection of portfolio sets $Z^i \subset \mathbb{R}^J$ for every agent $i \in I$,

The payoff matrix is given by the $(D \times J)$ matrix $V = (v(\xi, j))_{\xi \in \mathcal{D}, j \in \mathcal{J}}$, and satisfies the condition $v(\xi, j) = 0$ if $\xi \not\in \mathcal{D}^+(\xi(j))$.

The full matrix of payoffs $W_\mathcal{F}(q)$ is the $(D \times J)$–matrix with entries

$$w_\mathcal{F}(q)(\xi, j) := v(\xi, j) - \delta_{\xi, \xi(j)}q_j,$$
where $\delta_{\xi, \xi'} = 1$ if $\xi = \xi'$ and $\delta_{\xi, \xi'} = 0$ otherwise.

So, for a given portfolio $z \in \mathbb{R}^J$ (and asset price $q$) the full flow of returns is $W_F(q)z$ and the (full) financial return at node $\xi$ is

$$\begin{align*}
[W_F(q)z](\xi) &= W_F(q, \xi) \cdot z = \sum_{j \in J} v(\xi, j)z_j - \sum_{j \in J} \delta_{\xi, \xi(j)} q_j z_j \\
&= \sum_{\{j \in J| \xi(j) < \xi\}} v(\xi, j)z_j - \sum_{\{j \in J| \xi(j) = \xi\}} q_j z_j,
\end{align*}$$

and we shall extensively use the fact that, for $\lambda \in \mathbb{R}^D$, and $j \in J$, one has:

$$|W_F(q)\lambda|(j) = \sum_{\xi \in D} \lambda(\xi) v(\xi, j) - \sum_{\xi \in \mathbb{D}} \lambda(\xi) \delta_{\xi, \xi(j)}$$

(2.1)

In the following, when the financial structure $F$ remains fixed, while only prices vary, we shall simply denote by $W(q)$ the full matrix of returns. In the case of unconstrained portfolios, namely $Z^i = \mathbb{R}^I$, for every $i \in I$, the financial asset structure will be simply denoted by $F = (J, (\xi(j), V^i)_{j \in J})$.

### 2.4 Financial Equilibrium

We now consider a financial exchange economy, which is defined as the couple of an exchange economy $E$ and a financial structure $F$. It can thus be summarized by

$$(E, F) := [D, H, I, (X^i, P^i, e^i)_{i \in I}; J, (\xi(j), V^j)_{j \in J}, (Z^i)_{i \in I}].$$

Given the price $(p, q) \in \mathbb{R}^L \times \mathbb{R}^J$, the budget set of consumer $i \in I$ is\(^4\)

$$B_F^i(p, q) = \{(x^i, z^i) \in X^i \times Z^i : \forall \xi \in D, \ p(\xi) \bullet_H [x^i(\xi) - e^i(\xi)] \leq [W_F(q)z^i](\xi)\}$$

$$= \{(x^i, z^i) \in X^i \times Z^i : p \bullet (x^i - e^i) \leq W_F(q)z^i\}.$$

When $F$ is fixed we can drop the subscript $F$ from the budget set. We now introduce the equilibrium notion.

**Definition 2.2** An equilibrium of the financial exchange economy $(E, F)$ is a list of strategies and prices $(\bar{x}, \bar{z}, \bar{\rho}, \bar{q}) \in (\mathbb{R}^{L})^I \times (\mathbb{R}^{J})^I \times \mathbb{R}^{L} \setminus \{0\} \times \mathbb{R}^J$ such that

\(^4\)For $x = (x(\xi))_{\xi \in D}, p = (p(\xi))_{\xi \in D}$ in $\mathbb{R}^C$ (with $x(\xi), p(\xi)$ in $\mathbb{R}^E$) we let $p \bullet x = (p(\xi) \bullet x(\xi))_{\xi \in D} \in \mathbb{R}^D$. 


Definition 2.3 Given the financial structure $\mathcal{F} = (J, (\xi(j), V^j))_{j \in J}, (Z^i)_{i \in I}$, the portfolio $z^i \in Z^i$ is said to have no arbitrage opportunities or to be arbitrage-free for agent $i \in I$ at the price $q \in \mathbb{R}^J$ if there is no portfolio $z^i \in Z^i$ such that $W_\mathcal{F}(q)z^i > W_\mathcal{F}(q)\bar{z}^i$, that is, $[W_\mathcal{F}(q)z^i](\xi) \geq [W_\mathcal{F}(q)\bar{z}^i](\xi)$, for every $\xi \in D$, with at least one strict inequality, or, equivalently, if

$$W_\mathcal{F}(q) (Z^i - \bar{z}^i) \cap \mathbb{R}^D_+ = \{0\}.$$ 

We say $q$ is an arbitrage free asset price or the financial structure $\mathcal{F}$ is said to be arbitrage-free at $(q)$ if there exists no portfolios $z^i \in Z^i$ ($i \in I$) such that $W_\mathcal{F}(q)(\sum_{i \in I} z^i) > 0$, or, equivalently, if:

$$W_\mathcal{F}(q)\left(\sum_{i \in I} Z^i\right) \cap \mathbb{R}^D_+ = \{0\}.$$

Let the financial structure $\mathcal{F}$ be arbitrage-free at $q$, and let $z^i \in Z^i$ ($i \in I$) such that $\sum_{i \in I} W_\mathcal{F}(q)z^i = 0$, then it is easy to see that, for every $i \in I$, $z^i$ is arbitrage-free at $q$. The converse is true, when $\sum_{i \in I} W_\mathcal{F}(q)z^i \subset \text{cone } [\bigcup_{i \in I} W(q)(Z^i - \bar{z}^i)]$. The later is true in particular when some agent’s portfolio set is unconstrained, that is, $Z^i = \mathbb{R}^J$ for some $i \in I$.

Consider the following non-satiation assumption:

Assumption NS (i) For every $\bar{x} \in \prod_{i \in I} X^i$ such that $\sum_{i \in I} \bar{x}^i = \sum_{i \in I} e^i$,

(Non-Satiation at Every Node) for every $\xi^i \in D$, there exists $x \in \prod_{i \in I} X^i$ such that, for each $\xi \neq \xi^i$, $x^i(\xi) = \bar{x}^i(\xi)$ and $x^i \in P^i(\bar{x})$;

(ii) if $x^i \in P^i(\bar{x})$, then $[x^i, \bar{x}^i] \subset P^i(\bar{x})$.

It is well known that if preferences are non-satiated then there is no arbitrage at the individual level. In particular, under (NS), if $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium of the economy $(\mathcal{E}, \mathcal{F})$, then $z^i$ is arbitrage-free at $\bar{q}$ for every $i \in I$ (see Angeloni and Cornet [1]).
However, the set of asset prices at which there is no arbitrage at the individual level is larger than the set of asset prices that do not offer arbitrage opportunities at the aggregate level. We thus show that under some conditions any no-arbitrage price at the aggregate level can be supported as an equilibrium price.

3 Existence of Equilibrium

3.1 The Main Existence Result

We will prove that when agents’ portfolio sets are constrained, any arbitrage free asset price can be characterized as an equilibrium asset price. Our approach however does not cover the general case of real assets which needs a different treatment. Let us consider, the financial economy

\[(E, F) = [D, H, I, (X^i, P^i, e^i)_{i \in I}; J, (\xi(j), V^j)_{j \in J}, (Z^i)_{i \in I}].\]

Define the set of attainable consumptions by

\[
\hat{X} = \{ x \in \prod_{i \in I} X^i | \sum_{i \in I} x^i = \sum_{i \in I} e^i \}
\]

and for each \(i \in I\), let \(\hat{X}^i\) be the projection of \(\hat{X}\) on \(X^i\).

We introduce the following assumptions.

Assumption (C) (Consumption Side) For all \(i \in I\) and all \(\bar{x} \in \prod_{i \in I} X^i\),

(i) \(X^i\) is a closed and convex subset of \(\mathbb{R}^L\) and \(\hat{X}^i\) is compact\(^5\) in \(\mathbb{R}^L\);

(ii) the preference correspondence \(P^i : \prod_{i \in I} X^i \rightarrow X^i\), is lower semicontinuous\(^6\) and \(P^i(\bar{x})\) is convex;

(iii) for every \(x^i \in P^i(\bar{x})\) for every \((x')^i \in X^i, (x')^i \neq x^i, [(x')^i, x^i] \cap P^i(\bar{x}) \neq \emptyset;\)

(iv) (Irreflexivity) \(\bar{x}^i \not\in P^i(\bar{x});\)

(v) (Non-Satiation of Preferences at Every Node) if \(\sum_{i \in I} \bar{x}^i = \sum_{i \in I} e^i\), for every \(\xi \in D\) there exists \(x \in \prod_{i \in I} X^i\) such that, for each \(\xi' \neq \xi, x^i(\xi') = \bar{x}^i(\xi')\) and \(x^i \in P^i(\bar{x});\)

\(^5\)Note: \(\hat{X}^i\) is compact if \(X^i\) is bounded below.

\(^6\)A correspondence \(\varphi : X \rightarrow Y\) is said to be lower semicontinuous at \(x_0 \in X\) if, for every open set \(V \subset Y\) such that \(V \cap \varphi(x_0)\) is not empty, there exists a neighborhood \(U\) of \(x_0\) in \(X\) such that, for all \(x \in U, V \cap \varphi(x)\) is nonempty. The correspondence \(\varphi\) is said to be lower semicontinuous if it is lower semicontinuous at each point of \(X\).

\(^7\)This is satisfied, in particular, when \(P^i(\bar{x})\) is open in \(X^i\) (for its relative topology).
Strong Survival Assumption \( e^i \in \text{int} X^i \).

Assumption (F) (Financial Side) Given an asset price \( q \in \mathbb{R}^J \),

(i) for every \( i \in \mathcal{I} \), \( W(q)Z^i \) is a closed, convex subset of \( \mathbb{R}^D \) containing 0;

We can now state the main theorem characterizing equilibrium prices with arbitrage free prices under the appropriate compatibility condition.

**Theorem 3.1** Suppose the financial exchange economy \((\mathcal{E}, \mathcal{F})\) satisfies \( \mathbf{C} \) and \( \mathbf{F} \). Let \( \bar{q} \in \mathbb{R}^J \) satisfy the following conditions:

(i) \( (AF 2): \) Arbitrage-free at \( \bar{q} \)

\( W(\bar{q})(\sum_{i \in \mathcal{I}} Z^i) \cap \mathbb{R}^D_+ = \{0\} \)

(ii) \( (W 6): \)

\[ \text{Span} \left( W(\bar{q})(\sum_{i \in \mathcal{I}} Z^i) \right) \subset \text{cone} \left( \bigcup_{i \in \mathcal{I}} W(\bar{q})(Z^i) \right) \]

(iii) \( (WEQ) \)

\[ -\sum_{i \in \mathcal{I}} Z^i \cap \text{Ker} W(\bar{q}) \subset \sum_{i \in \mathcal{I}} [A(Z^i) \cap \text{Ker} W(\bar{q})] \]

Then there exists \((\bar{x}, \bar{z}, \bar{p})\) with \( \bar{p}(\xi) \neq 0, \forall \xi \in \mathcal{D} \) such that \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) is an equilibrium.

### 3.2 Existence of a Weak Equilibrium

The above result will be proved as a consequence of the following more general result, which is interesting by itself.

**Definition 3.1** A weak equilibrium in the economy \((\mathcal{E}, \mathcal{F})\) is a list \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) satisfying condition (a) and (b) in Definition 2.2 and the following:

\( (c') \)

\[ \sum_{i \in \mathcal{I}} W(\bar{q})Z^i = 0. \]

**Theorem 3.2** Suppose the financial exchange economy \((\mathcal{E}, \mathcal{F})\) satisfies \( \mathbf{C} \) and \( \mathbf{F} \). Let \( \bar{q} \in \mathbb{R}^J \) satisfy the following conditions:

(i) \( (AF 2): \) Arbitrage-free at \( \bar{q} \)

\( W(\bar{q})(\sum_{i \in \mathcal{I}} Z^i) \cap \mathbb{R}^D_+ = \{0\} \)

(ii) \( (W 6): \)

\[ \text{Span} \left( W(\bar{q})(\sum_{i \in \mathcal{I}} Z^i) \right) \subset \text{cone} \left( \bigcup_{i \in \mathcal{I}} W(\bar{q})(Z^i) \right) \]

Then there exists \((\bar{x}, \bar{z}, \bar{p})\) with \( \bar{p}(\xi) \neq 0, \forall \xi \in \mathcal{D} \) such that \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) is a weak equilibrium.

The proof of Theorem 3.1 is then a consequence of the following proposition due to Da-Rocha and Triki [6].

---

8Given a convex set \( Y \subset \mathbb{R}^n \), the asymptotic cone of \( Y \) is \( A(Y) := \{ t \in \mathbb{R}^n \mid y + t \in Y, \forall y \in Y \} \).
Proposition 3.1 Existence of a weak equilibrium \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) implies the existence of an equilibrium if we have:

\[
[-\sum_{i\in I} Z_i] \cap \text{Ker } W(\bar{q}) \subset \sum_{i\in I} [A(Z_i) \cap \text{Ker } W(\bar{q})]
\]

The condition in Proposition 3.1 holds if the following is true:

\[
\text{Ker } W \subset \bigcup_{i\in I} A(Z_i)
\]

3.3 Concept of No-arbitrage at the aggregate Level

Various notions of arbitrage free asset prices, found in the literature are listed below.

Given an asset price \(\bar{q} \in \mathbb{R}^J\):

(AF 1) \(\text{Im } (W(\bar{q})) \cap \mathbb{R}^D_+ = \{0\}\)

(AF 2) \(W(\bar{q})(\sum_{i\in I} Z_i) \cap \mathbb{R}^D_+ = \{0\}\)

(AF 3) \(W(\bar{q})A(\sum_{i\in I} Z_i)) \cap \mathbb{R}^D_+ = \{0\}\)

(AF 4) \(W(\bar{q})(\bigcup_{i\in I} A(Z_i)) \cap \mathbb{R}^D_+ = \{0\}\)

If there were some agent with an unconstrained portfolio set then all the above notions coincide and (AF 1) would suffice to describe absence of arbitrage at the aggregate level. Magill and Quinzii [15] use this notion. Da-Rocha and Triki [6] say that the payoff operator is arbitrage free under (AF 1).

Under constrained portfolio sets AF 2 would be more a more appropriate notion of absence of arbitrage in the aggregate level. This notion, due to Angeloni and Cornet [1], is considered in this paper.

The conditions AF 3 and AF 4 are weaker notions of absence of arbitrage. AF 3 says that there is no infinite arbitrage in the aggregate level. AF 4 says that no agent by herself can find an infinite arbitrage at \(\bar{q}\). Da-Rocha and Triki [6] say that financial markets are arbitrage free under this condition. The relationship between these conditions is given by the following proposition, the proof of which is immediate.

Proposition 3.2 Given \(\bar{q} \in \mathbb{R}^J\), we have the following:

\[AF 1 \implies AF 2 \implies AF 3 \implies AF 4\]

There are several notions of compatibility in terms of the portfolios sets that have been assumed in order to prove existence of an equilibrium. These are listed below. Given \(\bar{q} \in \mathbb{R}^J\):
For the above conditions each condition with a subscript \( s \) is a stronger version of the condition without the subscript. For instance \( W 2s \) is stronger \( W 2 \), and so on. Moreover the following proposition explains the relationship between these conditions, the proof of which is immediate.

\[ \text{Proposition 3.3} \quad \text{Given } \bar{q} \in \mathbb{R}^I \text{ we have the following:} \]

(a) \( W 1s \implies W 2s \implies W 3s \implies W 6s \)

(b) \( W 1s \implies W 4s \implies W 5s \implies W 6s \)

(c) \( W 1 \implies W 2 \implies W 3 \implies W 6 \)

(d) \( W 1 \implies W 4 \implies W 5 \implies W 6 \)

In order to show that any asset price at which \( (AF \ 4) \) (the most general notion of no-arbitrage) holds, can be supported as and equilibrium asset price, a stronger compatibility condition is required. The following proposition explains the reason for this.
Proposition 3.4 Given $\bar{q} \in \mathbb{R}^J$, we have the following:

(i) (Da-Rocha and Triki [6]) $\text{AF } 4$ and $\text{W } 2 \implies \text{AF } 1$ and $\text{W } 5$

(ii) $\text{AF } 4$ and $\text{W } 3 \implies \text{AF } 2$ and $\text{W } 6$

Proof of Proposition 3.4 (ii):

By Proposition 3.3 we have $\text{W } 3 \implies \text{W } 6$. By contradiction suppose $\text{AF } 3$ and $\text{W } 3$ hold but $\text{W } 6$ does not hold. Then there exists $w \in \text{Span } W\sum_{i \in I} Z_i$ such that $w > 0$. For all $t \in \mathbb{N}, tw \in \text{Span } W\sum_{i \in I} Z_i \subset \bigcup_{i \in I} WZ_i$. Since the set of agents is finite, there exists $i \in I$ and an increasing sequence of integers $(t_n)_n$ such that $t_nw \in WZ_i$ for each $n \in \mathbb{N}$. Let $w^i \in WZ_i$ then

$$ (1 - \frac{1}{t_n})w^i + \frac{1}{t_n}t_nw \in WZ_i $$

Passing to the limit we have $w^i + w \in WZ_i$. Thus $w \in A(WZ_i)$. Contradiction with $\text{AF } 3$. $\square$

3.4 Some Consequences of the Main Theorem

In view of Proposition 3.2, Proposition 3.3 and Proposition 3.4 we have the following consequences.

Corollary 3.1 (Unrestricted Case 1) If the conditions in Theorem 3.2 are replaced with:

(AF 2) [Arbitrage-free at $\bar{q}$] $W(\bar{q})(\sum_{i \in I} Z_i) \cap \mathbb{R}^D_+ = \{0\}$

(W 1s) $\exists i \in I$ such that $Z_i = \mathbb{R}^J$

Then there exists $(\bar{x}, \bar{z}, \bar{p})$ with $\bar{p}(\xi) \neq 0, \forall \xi \in D$ such that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a weak equilibrium.

Corollary 3.2 (Angeloni-Cornet) If the conditions in Theorem 3.2 are replaced with:

(AF 2) [Arbitrage-free at $\bar{q}$] $W(\bar{q})(\sum_{i \in I} Z_i) \cap \mathbb{R}^D_+ = \{0\}$

(W 4s) $\exists i \in I$ such that $0 \in \text{int } (Z_i)$

Then there exists $(\bar{x}, \bar{z}, \bar{p})$ with $\bar{p}(\xi) \neq 0, \forall \xi \in D$ such that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a weak equilibrium.

Corollary 3.3 If the conditions in Theorem 3.2 are replaced with:

(AF 2) [Arbitrage-free at $\bar{q}$] $W(\bar{q})(\sum_{i \in I} Z_i) \cap \mathbb{R}^D_+ = \{0\}$

(W 6s) $\text{Span } (\sum_{i \in I} Z_i) \subset \text{cone } [\bigcup_{i \in I} (Z_i)]$

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Then there exists \((\bar{x}, \bar{z}, \bar{p})\) with \(\bar{p}(\xi) \neq 0, \forall \xi \in \mathcal{D}\) such that \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) is a weak equilibrium.

**Corollary 3.4 (DaRocha-Triki 1)** If the conditions in Theorem 3.2 are replaced with:

(AF 1) [Arbitrage-free at \(\bar{q}\)] \(\text{Im} (W(\bar{q})) \cap \mathbb{R}^D_+ = \{0\}\)

(W 5) \(\text{Im} (W(\bar{q})) \subset \text{cone} [\bigcup_{i \in I} W(\bar{q})(Z^i)]\)

Then there exists \((\bar{x}, \bar{z}, \bar{p})\) with \(\bar{p}(\xi) \neq 0, \forall \xi \in \mathcal{D}\) such that \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) is a weak equilibrium.

Following are some more consequences when the more general notion of no-arbitrage (AF 4) is used. Notice that the compatibility conditions are stronger.

**Corollary 3.5 (Unrestricted Case 2)** If the conditions in Theorem 3.2 are replaced with:

(AF 4) [Arbitrage-free at \(\bar{q}\)] \(W(\bar{q})(\bigcup_{i \in I} A(Z^i)) \cap \mathbb{R}^D_+ = \{0\}\)

(W 1s) \(\exists i \in I\) such that \(Z^i = \mathbb{R}^j\)

Then there exists \((\bar{x}, \bar{z}, \bar{p})\) with \(\bar{p}(\xi) \neq 0, \forall \xi \in \mathcal{D}\) such that \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) is a weak equilibrium.

**Corollary 3.6 (Da-Rocha and Triki 2)** If the conditions in Theorem 3.2 are replaced with:

(AF 4) [Arbitrage-free at \(\bar{q}\)] \(W(\bar{q})(\bigcup_{i \in I} A(Z^i)) \cap \mathbb{R}^D_+ = \{0\}\)

(W 2) \(\text{Im} (W(\bar{q})) \subset \bigcup_{i \in I} W(\bar{q})(Z^i)\)

Then there exists \((\bar{x}, \bar{z}, \bar{p})\) with \(\bar{p}(\xi) \neq 0, \forall \xi \in \mathcal{D}\) such that \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) is a weak equilibrium.

**Corollary 3.7** If the conditions in Theorem 3.2 are replaced with:

(AF 4) [Arbitrage-free at \(\bar{q}\)] \(W(\bar{q})(\bigcup_{i \in I} A(Z^i)) \cap \mathbb{R}^D_+ = \{0\}\)

(W 3) \(\text{Span} (W(\bar{q})(\sum_{i \in I} Z^i)) \subset \bigcup_{i \in I} W(\bar{q})(Z_i)\)

Then there exists \((\bar{x}, \bar{z}, \bar{p})\) with \(\bar{p}(\xi) \neq 0, \forall \xi \in \mathcal{D}\) such that \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) is a weak equilibrium.
3.5 Counterexample 1:

It is possible to find a meaningful example where (AF 2) and (W 6) does not imply (AF 1) and (W 5). Thus the set of asset prices used to characterize weak equilibrium asset prices in Theorem 3.2 is larger than that in Da-Rocha and Triki [6].

Let $T = \{0, 1\}$, $D = \{0, 1, 2\}$ and $I = \{1, 2\}$. Let $J = \{1, 2\}$ such that for all $j \in J$, $\xi(j) = 0$. The payoffs are given by $V^1 = (2, 1)$ and $V^2 = (2, 2)$. Consider the asset price $\bar{q} = (2, 2)$. Then the total payoff matrix is given by:

$$W(\bar{q}, V) = \begin{bmatrix} -2 & -2 \\ 2 & 2 \\ 1 & 2 \end{bmatrix}$$

Let

$$Z^1 = Z^2 = \{\alpha(1, 1) | \alpha \in [-1, 1]\}$$

then

$$W(\bar{q}, V)Z^1 = W(\bar{q}, V)Z^2 = \{\alpha(-4, 4, 3) | \alpha \in [-1, 1]\}$$

$$W(\bar{q}, V)(\sum_{i \in I} Z^i) = \{\alpha(-8, 8, 6) | \alpha \in [-1, 1]\}$$

$$\text{span} \left[ W(\bar{q}, V)(\sum_{i \in I} Z^i) \right] = \text{cone} \left( \cup_{i \in I} W(\bar{q}, V)Z^i \right)$$

At this price $\bar{q}$, (AF 2) and (W 6) are satisfied but not (AF 1) and (W 5). This price $\bar{q}$ can be characterized as an equilibrium asset price according to Theorem 3.2.

For instance suppose there is only one good available in each state and the two agents have the following endowments and utility functions:

$$e^1 = (2, 6, 6) \text{ and } u^1(x^1(0), x^1(1), x^1(2)) = 2x^1(0) + x^1(1) + x^1(2)$$

$$e^2 = (4, 4, 3) \text{ and } u^2(x^2(0), x^2(1), x^2(2)) = x^2(0) + 2x^2(1) + 2x^2(2)$$

Each agent $i$’s income transfer potential is determined by $\alpha^i \in [-1, 1]$. After incorporating the budget constraints, we can then write the reduced from utility functions in terms of the choice of portfolio $\alpha^i$.

$$\hat{u}^1(\alpha^1) = 16 - \alpha^1$$
\[ u^2(\alpha^2) = 18 + 10\alpha^2 \]

The agents can then be viewed as maximizing their reduced form utilities over their choice of \( \alpha^i \in [-1, 1] \). So \( \alpha^1 = -1 \) maximizes agent 1’s utility and \( \alpha^2 = 1 \) maximizes agent 2’s utility.

Thus \( \bar{x}^1 = (6, 2, 3); \; \bar{z}^1 = (-1, -1) \) and \( \bar{x}^2 = (0, 8, 6); \; \bar{z}^2 = (1, 1) \) along with \( \bar{q} = (2, 2) \) is an equilibrium. \( \Box \)

Modifying the above example so that the compatibility requirement (W 6) does not hold, the impossibility of the characterization result is given in the following example.

### 3.6 Counterexample 2:

Here we show that the conditions in Theorem 3.2 are fairly tight. That is if AF 2 and W 6 do not hold at some price \( \bar{q} \), then \( \bar{q} \) cannot be an equilibrium asset price.

Consider the example above with the following modification of the agents’ portfolio sets:

\[ Z^1 = Z^2 = \{ \alpha(1, 1) \mid \alpha \in [0, 1] \} \]

then

\[ W(\bar{q}, V)Z^1 = W(\bar{q}, V)Z^2 = \{ \alpha(-4, 4, 3) \mid \alpha \in [0, 1] \} \]

\[ W(\bar{q}, V)(\sum_{i \in \mathcal{I}} Z^i) = \{ \alpha(-8, 8, 6) \mid \alpha \in [0, 1] \} \]

The same price in the previous example \( \bar{q} \) does not provide arbitrage opportunities at the aggregate level, since

\[ W(\bar{q}, V)(\sum_{i \in \mathcal{I}} Z^i) \cap \mathbb{R}_+^D = \{0\} \]

However the compatibility condition does not hold, since

\[ \text{span } W(\bar{q}, V)(\sum_{i \in \mathcal{I}} Z^i) = \{ \alpha(-4, 4, 3) \mid \alpha \in \mathbb{R} \} \]

is not contained in

\[ \text{cone } \bigcup_{i \in \mathcal{I}} W(\bar{q}, V)Z^i = \{ \alpha(-4, 4, 3) \mid \alpha \in \mathbb{R}_+ \} \]
Again working with the reduced form utility functions as in the previous example, note that for accounts clearing in the asset market at $\bar{q}$ we need $\alpha_1 = -\alpha_2$. Given $Z^1$ and $Z^2$, this is possible only when $\alpha_1 = \alpha_2 = 0$. However, from agent 2’s problem in the previous example we see that $\alpha_2 = 1$ is feasible within her budget restrictions and maximizes her reduced form utility. Thus $q = (2, 2)$ cannot be an equilibrium asset price.

4 Proof of Main Theorem

4.1 Proof of Theorem 3.2 (under additional assumptions):

Additional assumptions ($K$)

(i) The sets $X^i$ and $W(\bar{q})Z^i$ are bounded;

(ii) [Local Non-Satiation] for every $\bar{x} \in \prod_{i \in I} X^i$, for every $x^i \in P^i(\bar{x})$ then $[x^i, \bar{x}] \subset P^i(\bar{x})$.

Before entering the proof of Theorem 3.2 we will state and prove the following:

Lemma 4.1 If Conditions (i) and (ii) in Theorem 3.2 hold at $\bar{q}$ then under Assumption ($K$) there exists $\lambda \in \mathbb{R}^D_{++}$ such that:

$$W(\bar{q})(\sum_{i \in I} Z^i) \subset \lambda^\perp := \{t \in \mathbb{R}^D | \lambda \bullet_D t = 0\}$$

Proof of Lemma 4.1: For all $i \in I, W(\bar{q})(Z^i)$ is compact (by $F$ and $K(i)$). Thus $W(\bar{q})(\sum_{i \in I} Z^i)$ is compact and hence closed. Since $F$ is arbitrage free (condition (i) in theorem) at $\bar{q}$, there exists $\lambda \in \mathbb{R}^D_{++}$ such that

$$\forall w \in \sum_{i \in I} W(\bar{q})Z^i, \lambda \bullet_D w \leq 0.$$ 

Since $\forall i \in I, 0 \in W(\bar{q})Z^i$, we have $\forall w^i \in W(\bar{q})Z^i, \lambda \bullet_D w^i \leq 0$. By (ii) $\forall w \in W(\bar{q})(\sum_{i \in I} Z^i), -w \in cone \left[ \bigcup_{i \in I} W(\bar{q})Z^i \right]$. Thus $\exists k \in I, \exists \alpha > 0$ such that $-w = \alpha w^k$ for some $w^k \in W(\bar{q})Z^k$. Thus $-\lambda \bullet_D w = \lambda \bullet_D (\alpha w^k) \leq 0$.

To simplify the notation, in the following we will suppress the dependence of $W$ on $\bar{q}$.

Preliminaries

Define the following$^9$:

$$B = \{p \in \mathbb{R}^L | \|\lambda \bullet_D p\| \leq 1\}$$

$^9$For $x \in \mathbb{R}^n, \|x\|$ denotes the euclidean norm.
\[
\rho(p) = 1 - ||\lambda \Box p||
\]

Let \( I_1 = (1, \ldots , 1) \) denote the element in \( \mathbb{R}^D \), whose coordinates are all equal to one.

Let \( \Gamma \) denote the space of continuous functions from \( B \) to \( \mathbb{R}^D \). For every \( \gamma \in \Gamma \) we have \( \gamma = (\gamma(p, \xi))_{\xi \in D} \).

Given \( p \in B \) and \( \gamma \in \Gamma \), for all \( i \in I \) define:

\[
\beta_i^\gamma(p) = \left\{ (x^i, w^i) \in X^i \times W Z^i : \exists \tau^i \in [0, 1], p \Box (x^i - e^i) \leq w^i + \tau_i \gamma(p) + \rho(p) I_1 \right\},
\]

\[
\alpha_i^\gamma(p) = \left\{ (x^i, w^i) \in X^i \times W Z^i : \exists \tau^i \in [0, 1], p \Box (x^i - e^i) \ll w^i + \tau_i \gamma(p) + \rho(p) I_1 \right\}.
\]

Using the procedure outlined in Da-Rocha and Triki [6] we choose \( \gamma \in \Gamma \) as in the following lemma, and drop the subscript \( \gamma \) from the above sets \( \beta_i(p) \) and \( \alpha_i(p) \).

**Lemma 4.2** There exists a continuous mapping \( \gamma : B \rightarrow \mathbb{R}^D \) such that:

1. \( \forall p \in B, \lambda \Box \gamma(p) = 0 \)
2. \( \forall p \in B, \forall w \in W(\sum_{i \in I} Z^i), w \Box \gamma(p) = 0 \)
3. \( \forall p \in B, \bigcup_{i \in I} \alpha_i^\gamma(p) \neq \emptyset \)

**Proof of Lemma 4.2:** Define the following subsets of \( \Gamma \):

\[
\Gamma_1 := \{ \gamma \in \Gamma | \forall p \in B, \gamma(p) \in \lambda^\perp \}
\]

\[
\Gamma_2 := \{ \gamma \in \Gamma | \forall p \in B, \gamma(p) \in [W(\sum_{i \in I} Z^i)]^\perp \}
\]

\[
\Gamma_3 := \{ \gamma \in \Gamma | \forall p \in B, \bigcup_{i \in I} \alpha_i^\gamma(p) \neq \emptyset \}
\]

We will show that \( \Gamma_1 \cap \Gamma_2 \cap \Gamma_3 \neq \emptyset \).

**Step(1):** \( \Gamma_1 \cap \Gamma_2 \neq \emptyset \)

Define the set

\[
\Delta = \{ \delta \in \mathbb{R}^D | \delta \in \lambda^\perp \cap [W(\sum_{i \in I} Z^i)]^\perp, ||\delta|| \leq 1 \}
\]

Notice from assumption C that \( \exists r > 0 \) such that \( \forall i \in I, U := B_L(0, r) \subset X^i \).

Define the correspondence \( \Psi \) from \( B \) to \( \Delta \) by

\[
\Psi(p) = \{ \delta \in \Delta | \exists u \in U, \exists w \in \text{Span} (W(\sum_{i \in I} Z^i)), p \Box u \ll w + \delta + \rho(p) I_1 \}
\]
We will show that there is a continuous selection $\gamma$ of $\Psi$ such that $\gamma \in \Gamma_1 \cap \Gamma_2$. In order to do this we will use the continuous selection result in Proposition 1.5.3, Florenzano [11].

Firstly, notice that $\forall \; p \in B, \Psi(p)$ is clearly convex valued. To see this, let $\delta_1 \in \Psi(p)$ and $\delta_2 \in \Psi(p)$, then

$$\exists \; u_1 \in U, \exists \; w_1 \in \text{Span} (W(\sum_{i \in I} Z^i)) : p \square u_1 << w_1 + \delta_1 + \rho(p)I$$

$$\exists \; u_2 \in U, \exists \; w_2 \in \text{Span} (W(\sum_{i \in I} Z^i)) : p \square u_2 << w_2 + \delta_2 + \rho(p)I$$

Then for all $\alpha \in [0, 1], (\alpha u_1 + (1 - \alpha)w_2) \in U, (\alpha w_1 + (1 - \alpha)w_2) \in \text{Span} (W(\sum_{i \in I} Z^i))$ and $(\alpha \delta_1 + (1 - \alpha)\delta_2) \in \Delta$, and

$$p \square (\alpha u_1 + (1 - \alpha)w_2) << (\alpha w_1 + (1 - \alpha)w_2) + (\alpha \delta_1 + (1 - \alpha)\delta_2) + \rho(p)I$$

Thus for all $p \in B, \Psi(p)$ is convex.

Also, notice that $\forall \; p \in B, \Psi(p) \not= \emptyset$. To see this, let $p \in B$,

Case (i) $\rho(p) > 0$: Let $u = 0$ and $w = 0$ then $\delta = 0 \in \Psi(p)$.

Case (ii) $\rho(p) = 0$: i.e. $\sum_{\xi \in D} \lambda(\xi)^2||p(\xi)||^2 = 1$. Since $\lambda >> 0, \exists \; \xi \in D$ such that $p(\xi) \not= 0$.

Thus $\exists \; u \in U$ such that $p \square u < 0$. Since $\lambda >> 0, \exists \; t \in \lambda^\perp$ such that $p \square u << t$.

Since $R^D = \text{Span} (W(\sum_{i \in I} Z^i)) + [W(\sum_{i \in I} Z^i)]^\perp$, there exists $w \in \text{Span} (W(\sum_{i \in I} Z^i))$ and $\delta \in [W(\sum_{i \in I} Z^i)]^\perp$ such that $t = w + \delta$. Using Lemma 4.1 we can see that,

$$0 = \lambda \bullet_D t = \lambda \bullet_D w + \lambda \bullet_D \delta = 0 + \lambda \bullet_D \delta = \lambda \bullet_D \delta$$

Thus $\delta \in \lambda^\perp$. Now we have

$$p \square u << w + \delta$$

For $\tau > 0$ small enough

$$p \square (\tau u) << \tau w + \tau \delta$$

with $\tau u \in U, \tau w \in \text{Span} (W(\sum_{i \in I} Z^i))$ and $||\tau \delta|| \leq 1$. Hence $\tau \delta \in \Psi(p)$. Thus $\Psi(p) \not= \emptyset$.

In view of Case (i) and Case (ii), $\forall \; p \in B, \Psi(p) \not= \emptyset$.

Now we can show that $\Psi$ is lower semicontinuous on $B$.

Denote the graph of $\Psi$ by,

$$G_{\Psi} : \{(p, \delta) \in B \times \Delta \mid \delta \in \Psi(p)\}$$

We will show $G_{\Psi}$ is open and hence $\Psi$ is l.s.c. In fact we will show $(B \times \Delta) \setminus G_{\Psi}$ is closed.
Let \{(p_n, \delta_n)\} ∈ (B × Δ) \setminus G_Ψ and (p_n, \delta_n) → (p, \delta). By contradiction suppose (p, \delta) ∈ G_Ψ.

Then \(\exists \bar{u} \in U, \exists \bar{w} \in \text{Span}(W(\sum_{i \in I} Z_i))\) such that

\[
∀ \xi \in D, \quad p(\xi) \cdot H \bar{u}(\xi) \ll \bar{w}(\xi) + \delta(\xi) + \rho(p) \tag{*}
\]

Also \(∀ n, \delta_n \notin Ψ(p_n)\) thus \(∀ n, \forall u \in U \text{ and } \forall w \in \text{Span}(W(\sum_{i \in I} Z_i))\)

\[
\exists \xi \in D \text{ such that } p_n(\xi) \cdot u(\xi) ≥ w(\xi) + \delta_n(\xi) + \rho(p)
\]

in particular,

\[
\exists \xi \in D \text{ such that } p_n(\xi) \cdot \bar{u}(\xi) ≥ \bar{w}(\xi) + \delta_n(\xi) + \rho(p)
\]

Since \(p_n → p, \rho(p_n) → \rho(p)\) and \(∀ \xi \in D, p_n(\xi) → p(\xi)\). Since \(\delta_n → \delta, ∀ \xi \in D, \delta_n(\xi) → \delta(\xi)\). Thus in the limit

\[
\exists \xi \in D \text{ such that } p(\xi) \cdot \bar{u}(\xi) ≥ \bar{w}(\xi) + \delta(\xi) + \rho(p)
\]

Contradiction with (*). Thus Ψ is l.s.c. on B.

Applying Proposition 1.5.3 in Florenzano [11] we can conclude that there is a continuous selection \(γ \) of Ψ and \(∀ p \in B, γ(p) ∈ λ^⊥ ∩ |W(\sum_{i \in I} Z_i)|^⊥\). Thus \(γ ∈ Π_1 \cap Π_2\).

**Step(2):** \(Π_1 \cap Π_2 \subset Π_3\)

Let \(γ\) be the continuous selection of Ψ obtained in Step (1) above. We need to show \(∀ p ∈ B, \exists k ∈ I \text{ such that } \alpha_k(p) \neq ∅\). To see this, let \(p ∈ B\).

Case(i) \(ρ(p) > 0\): for all \(i ∈ I \text{ with } τ^i = 0 \) we have \((x^i, w^i) = (e^i, 0) ∈ α^i_γ(p)\).

Case(ii) \(ρ(p) = 0\). Since \(γ(p) ∈ Ψ(p), \exists u ∈ U, \exists w ∈ \text{Span}(W(\sum_{i ∈ I} Z_i))\) such that

\[
p □ u \ll w + γ(p)
\]

for \(τ ∈ (0, 1)\) small enough

\[
p □ (τu) \ll τw + τγ(p)
\]

with \((τu) ∈ U, (τw) ∈ WZ_k\) for some \(k ∈ I\) (by Condition (ii) in Theorem 3.2). Thus there exists \(k ∈ I\) such that setting \(τ^k = τ\) we have \((τu + e^k, τw) ∈ α^k_γ(p)\). Thus \(γ ∈ Π_3\).

In view of Step (1) and Step (2) we have \(Π_1 \cap Π_2 \cap Π_3 \neq ∅\). □
4.1.1 The Fixed Point Argument

For \((x, w, p) \in \prod_{i \in I} X^i \times \prod_{i \in I} W Z^i \times B\), we define the correspondences \(\Phi^i\) for \(i \in I_0 = \{0\} \cup I\) as follows:

\[
\Phi^0(x, w, p) = \left\{ (p') \in B \mid \lambda \cdot \sum_{i \in I} (x^i - e^i) > 0 \right\},
\]

and for every \(i \in I\),

\[
\Phi^i(x, w, p) = \begin{cases} 
\{(e^i, 0)\} & \text{if } (x^i, w^i) \notin \beta^i(p) \text{ and } \alpha^i(p) = \emptyset, \\
\beta^i(p) & \text{if } (x^i, w^i) \notin \beta^i(p) \text{ and } \alpha^i(p) \neq \emptyset, \\
\alpha^i(p) \cap (P^i(x) \times W Z^i) & \text{if } (x^i, w^i) \in \beta^i(p).
\end{cases}
\]

The existence proof relies on the following fixed-point-type theorem due to Gale and MasCollel ([17]).

**Theorem 4.1** Let \(I_0\) be a finite set, let \(C^i (i \in I_0)\) be a nonempty, compact, convex subset of some Euclidean space, let \(C = \prod_{i \in I_0} C^i\) and let \(\Phi^i (i \in I_0)\) be a correspondence from \(C\) to \(C^i\), which is lower semicontinuous and convex-valued. Then, there exists \(\bar{c} \in C\) such that, for every \(i \in I_0\) [either \(\bar{c}^i \in \Phi^i(\bar{c})\) or \(\Phi^i(\bar{c}) = \emptyset\)].

We now show that, the set \(C^0 = B\), and for all \(i \in I, C^i = X^i \times Z^i\) and the above defined correspondences \(\Phi^i (i \in I_0)\) satisfy the assumptions of Theorem 4.1.

**Claim 4.1** For every \(\bar{c} := (\bar{x}, \bar{w}, \bar{p}, \cdot) \in \prod_{i \in I} X^i \times \prod_{i \in I} W Z^i \times B\),

(i) \(\Phi^i(\bar{c})\) is convex (possibly empty)

(ii) \(\bar{p} \notin \Phi^0(\bar{c}), \text{ and for all } i \in I, (\bar{x}^i, \bar{w}^i) \notin \Phi^i(\bar{c})\)

(iii) for every \(i \in I_0\), the correspondence \(\Phi^i\) is lower semicontinuous at \(\bar{c}\)

**Proof of Claim 4.1:** Let \(\bar{c} := (\bar{x}, \bar{y}, (\bar{p})) \in \prod_{i \in I} X^i \times \prod_{i \in I} W Z^i \times B\) be given.

**Proof of (i):** Clearly \(\Phi^0(\bar{c})\) is convex. For every \(i \in I\), recalling that \(P^i(\bar{x})\) and \(W Z^i\) are convex sets, by Assumption (C) and (F), we have \(\Phi(\bar{c})\) is a convex set.

**Proof of (ii):** Clearly, \((\bar{p}) \notin \Phi^0(\bar{c})\) and \((\bar{x}^i, \bar{w}^i) \notin \Phi^i(\bar{c})\) follows from the definitions of these sets and the fact that \(\bar{x}^i \notin \Phi^i(\bar{c})\) (from Assumption (C)).

**Proof of (iii):** We need to show \(\Phi^i\) is lower semicontinuous for all \(i \in I^0\). Since \(\Phi^0\) has an open graph, clearly it is lower semicontinuous. To show lower semicontinuity of \(\Phi^i\) for \(i \in I\), we will distinguish three cases:
Case (1): $(\bar{x}^i, \bar{w}^i) \notin \beta^i(\bar{p})$ and $\alpha^i(\bar{p}) = \emptyset$. Then $\Phi^i(\bar{c}) = \{(e^i, 0)\} \subset U$. The set $\Omega^i = \{(x^i, w^i, p) : (x^i, w^i) \notin \beta^i(p)\}$ is an open subset of $X^i \times WZ^i \times B$ (by Assumptions (C) and (F)). To see this, let $\{(x_n, w_n, p_n)\}$ be such that $(x_n, w_n) \in \beta^i(p_n)$ and $(x_n, w_n, p_n) \to (x, w, p)$. Since for all $n, (x_n, w_n) \in \beta^i(p_n)$, there exists $\tau_n \in [0, 1]$ such that

$$p_n \square (x_n - e^i) \leq w_n + \tau_n \gamma(p_n) + \rho(p_n)1$$

In the limit we have

$$p \square (x - e^i) \leq w + \tau \gamma(p) + \rho(p)1$$

Where $\tau = \lim_{n \to \infty} \tau_n \in [0, 1]$. Thus $(x, w) \in \beta^i(p)$.

Thus $\Omega^i$ contains an open neighborhood $O$ of $\bar{c}$. Now, let $c = (x, w, p) \in O$. If $\alpha^i(p) = \emptyset$ then $\Phi^i(c) = \{(e^i, 0)\} \subset U$ and so $\Phi^i(c) \cap U$ is nonempty. If $\alpha^i(p) \neq \emptyset$ then $\Phi^i(c) = \beta^i(p)$.

But Assumptions (C) and (F) imply that $(e^i, 0) \in X^i \times WZ^i$, hence $(e^i, 0) \in \beta^i(p)$ (with $\tau^i = 0$ and noticing that $\rho(p) \geq 0$). So $\{(e^i, 0)\} \subset \Phi^i(c) \cap U$ which is also nonempty.

Case (2): $\bar{c} = (\bar{x}^i, \bar{w}^i, \bar{p}) \in \Omega^i := \{c = (x^i, w^i, p) : (x^i, w^i) \notin \beta^i(p) \text{ and } \alpha^i(p) \neq \emptyset\}$. Then the set $\Omega^i$ is clearly open (since its complement is closed).

On the set $\Omega^i$ one has $\Phi^i(c) = \beta^i(p)$. We recall that $\emptyset \neq \Phi^i(\bar{c}) \cap U = \beta^i(\bar{p}) \cap U$. We notice that $\beta^i(\bar{p}) = \text{cl } \alpha^i(\bar{p})$ since $\alpha^i(\bar{p}) \neq \emptyset$. Consequently, $\alpha^i(\bar{p}) \cap U \neq \emptyset$ and we chose a point $(x^i, w^i) \in \alpha^i(\bar{p}) \cap U$, that is, $(x^i, w^i) \in [X^i \times WZ^i] \cap U$ and for some $\tau^i \in [0, 1]$

$$\bar{p} \square (x^i - e^i) \ll w^i + \tau^i \gamma(\bar{p}) + \rho(\bar{p})1.$$ 

Clearly the above inequality is also satisfied for the same point $(x^i, w^i)$ and the same $\tau^i$ when $p$ belongs to a neighborhood $O$ of $\bar{p}$ small enough (using the continuity of $\rho(\cdot)$ and $\gamma(\cdot)$). This shows that on $O$ one has $\emptyset \neq \alpha^i(p) \cap U \subset \beta^i(p) \cap U = \Phi(c) \cap U$.

Case (3): $(\bar{x}^i, \bar{w}^i) \in \beta^i(\bar{p})$. By assumption we have

$$\emptyset \neq \Phi^i(\bar{c}) \cap U = \alpha^i(\bar{p}) \cap [P^i(\bar{x}) \times WZ^i] \cap U.$$ 

By an argument similar to what is done above, one shows that there exists an open neighborhood $N$ of $\bar{p}$ and an open set $M$ such that, for every $p \in N$, one has $\emptyset \neq M \subset \alpha^i(p) \cap U$.

Since $P^i$ is lower semicontinuous at $\bar{c}$ (by Assumption (C)), there exists an open neighborhood $\Omega$ of $\bar{c}$ such that, for every $c \in \Omega$, $\emptyset \neq [P^i(x) \times WZ^i] \cap M$, hence

$$\emptyset \neq [P^i(x) \times WZ^i] \cap \alpha^i(p) \cap U \subset \beta^i(p) \cap U, \text{ for every } c \in \Omega.$$ 

Consequently, from the definition of $\Phi^i$, we get $\emptyset \neq \Phi^i(c) \cap U$, for every $c \in \Omega$.

The correspondence $\Psi^i := \alpha^i \cap (P^i \times WZ^i)$ is lower semicontinuous on the whole set, being the intersection of an open graph correspondence and a lower semicontinuous
correspondence. Then there exists an open neighborhood $O$ of $\bar{c} := (\bar{x}, \bar{w}, \bar{p})$ such that, for every $(x, w, p) \in O$, then $U \cap \Psi^i(x, w, p) \neq \emptyset$ hence $\emptyset \neq U \cap \Phi^i(x, w, p)$ (since we always have $\Psi^i(x, w, p) \subset \Phi^i(x, w, p)$). □

In view of Claim 4.1, we can apply the Gale-MasColell theorem. Then we have the following:

\begin{equation}
\forall \bar{p} \in B, \ p \bigodot \sum_{i \in I} (\bar{x}^i - \bar{e}^i) \leq \bar{p} \bigodot \sum_{i \in I} (\bar{x}^i - e^i)
\end{equation}

\begin{equation}
\forall i \in I, (\bar{x}^i, \bar{w}^i) \in \beta^i(\bar{p}) \text{ and } \alpha^i(\bar{p}) \cap (P^i(\bar{x}) \times WZ^i) = \emptyset.
\end{equation}

\begin{equation}
\forall i \in I, \ \exists \bar{z}^i \in Z^i \text{ such that } \bar{w}^i = W(\bar{q})\bar{z}^i.
\end{equation}

### 4.1.2 The List $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a Weak Equilibrium

**Claim 4.2** $\sum_{i \in I}(\bar{x}^i - e^i) = 0$.

**Proof of Claim 4.2:**

Suppose $\sum_{i \in I}(\bar{x}^i - e^i) \neq 0$. From Assertion 4.1 we have

\begin{equation*}
\forall p \in B, \ p \bigodot \sum_{i \in I} (\bar{x}^i - e^i) \leq \bar{p} \bigodot \sum_{i \in I} (\bar{x}^i - e^i)
\end{equation*}

scalar product with $\lambda$ gives

\begin{equation*}
\forall p \in B, \ \ (\lambda \bigodot p) \bullet \sum_{i \in I} (\bar{x}^i - e^i) \leq (\lambda \bigodot \bar{p}) \bullet \sum_{i \in I} (\bar{x}^i - e^i)
\end{equation*}

Thus $(\lambda \bigodot \bar{p}) = \frac{\sum_{i \in I}(\bar{x}^i - e^i)}{||\sum_{i \in I}(\bar{x}^i - e^i)||}$ and $||\lambda \bigodot p|| = 1$. So $(\lambda \bigodot \bar{p}) \bullet \sum_{i \in I}(\bar{x}^i - e^i) > 0$.

From Assertion (4.2) $\forall i \in I, \exists \bar{r}^i \in [0, 1]$ such that

\begin{equation*}
p \bigodot (\bar{x}^i - e^i) \leq \bar{w}^i + \bar{r}^i \gamma(\bar{p}) + \rho(\bar{p})1
\end{equation*}

sum over $i$ to get:

\begin{equation*}
p \bigodot \sum_{i \in I}(\bar{x}^i - e^i) \leq \sum_{i \in I} \bar{w}^i + \sum_{i \in I} \bar{r}^i \gamma(\bar{p}) + \rho(\bar{p})1
\end{equation*}

taking scalar product with $\lambda$ we have the following:

\begin{equation*}
(\lambda \bigodot \bar{p}) \bullet \sum_{i \in I} (\bar{x}^i - e^i) \leq \lambda \bullet \sum_{i \in I} \bar{w}^i + \lambda \bullet \sum_{i \in I} \bar{r}^i \gamma(\bar{p}) + \rho(\bar{p})\lambda \cdot 1
\end{equation*}
On the RHS we have

\[ \lambda \bullet \sum_{i \in I} \bar{w}^i = 0 \quad (\text{by Lemma 4.1}) \]

\[ (\sum_{i \in I} \tilde{\gamma}_i) \lambda \bullet \gamma(\bar{p}) = 0 \quad (\text{since } \gamma \in \Gamma_1) \]

\[ \rho(\bar{p}) = 0 \quad (\text{since } ||\lambda \bullet \bar{p}|| = 1) \]

Thus \((\lambda \bullet \bar{p}) \bullet \sum_{i \in I}(\bar{x}^i - e^i) \leq 0\). Contradiction. \(\Box\)

**Claim 4.3** The following conditions hold:

(i) \(\forall \xi \in \mathbb{R}^D, \bar{p}(\xi) \neq 0\)

(ii) \(\forall i \in I, (\bar{x}^i, \bar{w}^i) \in \beta^i(\bar{p}) \) and \(\beta^i(\bar{p}) \cap (P^i(\bar{x}) \times WZ^i) = \emptyset\)

**Proof of Claim 4.3:** Since \(\gamma \in \Gamma_3, \exists k \in I\) such that \(\alpha^k(\bar{p}) \neq \emptyset\). We will first show Condition (ii) for consumer \(k\).

From Assertion (4.2) in the fixed point theorem, for consumer \(k\) one has \((\bar{x}^k, \bar{w}^k) \in \beta^k(\bar{p})\).

Now, suppose that \(\beta^k(\bar{p}) \cap (P^k(\bar{x}) \times WZ^k) \neq \emptyset\) and let \((\bar{x}^k, \bar{w}^k) \in \beta^k(\bar{p}) \cap (P^k(\bar{x}) \times WZ^k)\). Since \(\gamma \in \Gamma_3\) we have \(\alpha^k(\bar{p}) \neq \emptyset\) and we let \((\bar{x}^k, \bar{w}^k) \in \alpha^k(\bar{p})\).

Suppose first that \(\bar{x}^k = x^k\), then, from above \((\bar{x}^k, \bar{w}^k) \in [P^k(\bar{x}) \times WZ^k] \cap \alpha^k(\bar{p})\), which contradicts the fact that this set is empty by Assertion (4.2). Suppose now that \(\bar{x}^k \neq x^k\), from Assumption (C.iii), (recalling that \(x^k \in P^k(\bar{x})\)) the set \([\bar{x}^k, x^k] \cap P^k(\bar{x})\) is nonempty, hence contains a point \(x^k(\lambda) := (1 - \lambda)x^k + \lambda x^k\) for some \(\lambda \in [0, 1]\). We let \(w^k(\lambda) := (1 - \lambda)\bar{w}^k + \lambda w^k\) and we check that \((x^k(\lambda), w^k(\lambda)) \in \alpha^k(\bar{p})\) (since \((x^k, w^k) \in \beta^k(\bar{p})\) and \((\bar{x}^k, \bar{w}^k) \in \alpha^k(\bar{p})\)). Consequently, \(\alpha^k(\bar{p}) \cap (P^k(\bar{x}) \times WZ^k) \neq \emptyset\), which contradicts again Assertion (4.2).

Thus for agent \(k\) we have

\((\bar{x}^k, \bar{w}^k) \in \beta^k(\bar{p}) \) and \(\beta^k(\bar{p}) \cap (P^k(\bar{x}) \times WZ^k) = \emptyset \quad (**\)

**Proof of (i):** Suppose there exists \(\xi \in \mathcal{D}\) such that \(\rho(\xi) = 0\). From Claim 4.2, \(\sum_{i \in I} \bar{x}^i = \sum_{i \in I} \xi^i\), and from the Non-Satiation Assumption at node \(\xi\) (for Consumer \(k\)) there exists \(x^k \in P^k(\bar{x})\) such that \(x^k(\xi') = x^k(\xi')\) for every \(\xi' \neq \xi\); from Assertion (4.2), \((\bar{x}^k, \bar{w}^k) \in \beta^k(\bar{p})\) and, recalling that \(\bar{p}(\xi) = 0\), one deduces that \((x^k, w^k) \in \beta^k(\bar{p})\). Consequently,

\(\beta^k(\bar{p}) \cap [P^k(\bar{x}) \cap WZ^k] \neq \emptyset\),

which contradicts (**).
The following conditions hold: \( \forall \sum \rho \), which contradicts Claim 4.3. 

Proof of (i): We first prove that the modified budget constraints are binding, that is for all \( i \in I \) one has \( \bar{x}_i \in \beta^i(\bar{p}) \). From the Survival Assumption and the fact that \( \bar{p}(\xi) \neq 0 \) for every \( \xi \in \mathcal{D} \) (by Part (i) of this claim), one deduces that \( \alpha^i(\bar{p}) \neq \emptyset \). 

For each \( i \in I \setminus \{k\} \) we can repeating the same steps done in the beginning of the proof of this claim to get that for all \( i \in I \), \( (\bar{x}_i, \bar{w}_i) \in \beta^i(\bar{p}) \) and \( \beta^i(\bar{p}) \cap (P^i(\bar{x}) \times WZ^i) = \emptyset \). \( \square \)

**Claim 4.4** The following conditions hold:

(i) \( \rho(\bar{p}) = 0 \)

(ii) \( \sum_{i \in I} \bar{w}_i = 0 \)

(iii) \( \forall i \in I, \bar{\tau}_i \gamma(\bar{p}) = 0 \)

Proof of Claim 4.4:

Proof of (i): We first prove that the modified budget constraints are binding, that is for all \( i \in I \) we have the following assertion:

\[
\bar{p} \bigcirc (\bar{x}_i - e^i) = \bar{w}_i + \bar{\tau}_i \gamma(\bar{p}) + \rho(\bar{p}) \mathbf{1}
\]

Suppose not, then there exists \( i \in I \) such that

\[
\bar{p} \bigcirc (\bar{x}_i - e^i) < \bar{w}_i + \bar{\tau}_i \gamma(\bar{p}) + \rho(\bar{p}) \mathbf{1}
\]

That is there exist \( \xi \in \mathcal{D} \) such that

\[
\bar{p}(\xi) \bullet_H (\bar{x}_i(\xi) - e^i(\xi)) < \bar{w}_i(\xi) + \bar{\tau}_i \gamma(\bar{p})(\xi) + \rho(\bar{p})
\]

But by Claim 4.2, \( \sum_{i \in I} \bar{x}_i = \sum_{i \in I} e^i \) and by the nonsatiation assumption \( C(v) \) for consumer \( i \), there exists \( x_i \in P^i(\bar{x}) \) such that \( x_i(\xi') = \bar{x}_i(\xi') \) for every \( \xi' \neq \xi \). Consequently, we can choose \( x \in [x^i, \bar{x}_i] \) close enough to \( x_i \) so that \( (x, \bar{w}_i) \in \beta^i(\bar{p}) \). But, from the local nonsatiation (Assumption (K.ii)), \( [x^i, \bar{x}_i] \subset P^i(\bar{x}) \). Consequently, \( \beta^i(\bar{p}) \cap (P^i(\bar{x}) \times WZ^i) \neq \emptyset \) which contradicts Claim 4.3.

In Assertion 4.4 taking scalar product with \( \lambda \) yields:

\[
(\lambda \bigcirc \bar{p}) \bullet_L (\bar{x}_i - e^i) = \lambda \bullet_D \bar{w}_i + \bar{\tau}_i \lambda \bullet_D \gamma(\bar{p}) + \rho(\bar{p}) \lambda \bullet_L \mathbf{1}
\]

Since \( \gamma \in \Gamma_1 \), \( \lambda \bullet_D \gamma(\bar{p}) = 0 \). So we have,

\[
(\lambda \bigcirc \bar{p}) \bullet_H (\bar{x}_i - e^i) = \lambda \bullet_D \bar{w}_i + \rho(\bar{p}) \lambda \bullet_D \mathbf{1}
\]

\footnote{Take \( \bar{w}_i = 0, \bar{\tau}_i = 0 \) and \( \bar{x}_i = e^i - t\bar{p} \) for \( t > 0 \) small enough, so that \( \bar{x}_i \in X^i \) (from the Survival Assumption). Then notice that \( \bar{p} \bigcirc (\bar{x}_i - e^i) = -t(\bar{p} \bigcirc \bar{p}) \ll 0 \leq 0 + \rho(\bar{p}) \mathbf{1} \).}
Summing over $i \in I$, we have

$$(\lambda \otimes \bar{p}) \bullet \left( \sum_{i \in I} (\bar{x}^i - e^i) \right) = \lambda \bullet (\sum_{i \in I} \bar{w}^i) + I_\mathbb{P}(\bar{p}) \lambda \bullet 1$$

Since $\sum_{i \in I} \bar{w}^i \in W(\bar{q})(\sum_{i \in I} Z^i)$, by Lemma 4.1 and Claim 4.2 we have:

$$0 = I_\mathbb{P}(\bar{p}) \lambda \bullet 1$$

Thus $\rho(\bar{p}) = 0$. $\square$

**Proof of (ii) and (iii):** In Assertion (4.4) in view of Claim 4.4 (i), we have $\forall i \in I$:

$$\bar{p} \otimes (\bar{x}^i - e^i) = \bar{w}^i + \bar{p}^i \gamma(\bar{p})$$

Summing over $i \in I$ and using Claim 4.2 we have:

$$0 = \sum_{i \in I} \bar{w}^i + (\sum_{i \in I} \bar{p}^i) \gamma(\bar{p})$$

Since $\gamma \in \Gamma_2$ we have

$$0 = (\sum_{i \in I} \bar{p}^i) \gamma(\bar{p}) : (\sum_{i \in I} \bar{w}^i) = -(\sum_{i \in I} \bar{w}^i) : (\sum_{i \in I} \bar{p}^i)$$

Thus $\sum_{i \in I} w^i = 0$.

Since $\forall i \in I, \tau^i \geq 0$ and $(\sum_{i \in I} \bar{p}^i) \gamma(\bar{p}) = 0$ we have $\forall i \in I, \bar{p}^i \gamma(\bar{p}) = 0$. $\square$

**Claim 4.5** For every $i \in I$, $(\bar{x}^i, \bar{z}^i) \in B^i(\bar{p}, \bar{q})$ and $[P^i(\bar{x}) \times Z^i] \cap B^i(\bar{p}, \bar{q}) = \emptyset$.

**Proof of Claim 4.5:** In view of Claim 4.4 and Assertion 4.2, we have for all $i \in I$

$$p \otimes (\bar{x}^i - e^i) = \bar{w}^i$$

Thus in view of Assertion 4.3 $(\bar{x}^i, \bar{z}^i) \in B^i(\bar{p}, \bar{q})$.

Moreover by Assertion 4.2

$$\beta^i(\bar{p}) \cap [P^i(\bar{x}) \times WZ^i] = \emptyset$$

This condition along with Assertion 4.3 implies

$$B^i(\bar{p}, \bar{q}) \cap [P^i(\bar{x}) \times Z^i] = \emptyset$$

[Note that by Assertion 4.3 for all $i \in I$ there exists $\bar{z}^i \in Z^i$ such that $\bar{w}^i = W(\bar{q})\bar{z}^i$. Suppose $B^i(\bar{p}, \bar{q}) \cap [P^i(\bar{x}) \times Z^i] \neq \emptyset$. Then it contains a point $(x^i, z^i)$ such that $p \otimes (x^i - e^i) \leq W(\bar{q})\bar{z}^i$.

Letting $w^i = W(\bar{q})\bar{z}^i$ we have $(x^i, w^i) \in \beta^i(\bar{p})$. Also $x^i \in P^i(\bar{x})$. Thus $(x^i, w^i) \in [P^i(\bar{x}) \times WZ^i]$. Which implies $\beta^i(\bar{p}) \cap [P^i(\bar{x}) \times WZ^i] \neq \emptyset$. Contradiction with Assertion 4.2.] $\square$

In view of Claim 4.2, Assertion 4.3, Claim 4.3, Claim 4.4 and Claim 4.5 we have $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a weak equilibrium.
4.2 Proof in the General Case (without additional assumptions)

We now give the proof of Theorem 3.2, without considering the additional Assumption (K), as in the previous section. We will first enlarge the strict preferred sets as in Gale-Mas Colell, and then truncate the economy $E$ by a standard argument to define a new economy $\tilde{E}$, which satisfies all the assumptions of $E$, together with the additional Assumption (K).

From the previous section, there exists a weak equilibrium of $\tilde{E}$, and we will then check that it is also a weak equilibrium of $E$.

4.2.1 Enlarging the Preferences as in Gale-Mas Colell

The original preferences $P_i$ are replaced by the “enlarged” ones $\hat{P}_i$ defined as follows. For every $i \in I$, $\bar{x} \in \prod_{i \in I} X_i$ we let

$$\hat{P}_i(\bar{x}) := \bigcup_{x^i \in P_i(\bar{x})} \{ x^i + t(x^i - \bar{x}^i) \mid t \in ]0, 1], x^i \in P_i(\bar{x}) \}.$$  

The next proposition shows that $\hat{P}_i$ satisfies the same properties as $P_i$, for every $i \in I$, together with the additional Local Nonsatiation Assumption (K.ii).

**Proposition 4.1** Under (C), for every $i \in I$ and every $\bar{x} \in \prod_{i \in I} X_i$ one has:

(i) $P_i(\bar{x}) \subset \hat{P}_i(\bar{x}) \subset X_i$;

(ii) the correspondence $\hat{P}_i$ is lower semicontinuous at $\bar{x}$ and $\hat{P}_i(\bar{x})$ is convex;

(iii) for every $y^i \in \hat{P}_i(\bar{x})$ for every $(x')^i \in X_i, (x')^i \neq y^i$ then $[(x')^i, y^i] \cap \hat{P}_i(\bar{x}) \neq \emptyset$;

(iv) $\bar{x}^i \notin \hat{P}_i(\bar{x})$;

(v) (Non-Satiation at Every Node) if $\sum_{i \in I} \bar{x}^i = \sum_{i \in I} e^i$, for every $\xi \in D$, there exists $x \in \prod_{i \in I} X_i$ such that, for each $\xi' \neq \xi$, $x^i(\xi') = \bar{x}^i(\xi')$ and $x^i \in \hat{P}_i(\bar{x})$;

(vi) for every $y_i \in \hat{P}_i(\bar{x})$, then $[y^i, \bar{x}^i] \subset \hat{P}_i(\bar{x})$.

**Proof.** Let $\bar{x} \in \prod_{i \in I} X_i$ and let $i \in I$.

Part (i). It follows by the convexity of $X_i$, for every $i \in I$.

Part (ii). Let $y^i \in \hat{P}_i(\bar{x})$ and consider a sequence $(\bar{x}_n)_n \subset \prod_{i \in I} X_i$ converging to $\bar{x}$. Since $y^i \in \hat{P}_i(\bar{x})$, then $y^i = \bar{x}^i + t(x^i - \bar{x}^i)$ for some $x^i \in P_i(x^i)$ and some $t \in ]0, 1]$. Since $P^i$ is lower semicontinuous, there exists a sequence $(x^i_n)_n$ converging to $x^i$ such that $x^i_n \in P_i(x^i_n)$ for every $n \in \mathbb{N}$. Now define $y^i_n := \bar{x}^i_n + t(x^i_n - \bar{x}^i_n) \in [\bar{x}_n, x^i_n]$; then $y^i_n \in \hat{P}_i(\bar{x}_n)$ and obviously the sequence $(y^i_n)$ converges to $y^i$. This shows that $\hat{P}_i$ is lower semicontinuous at $\bar{x}$.
To show that $\hat{P}^i(\bar{x})$ is convex, let $y^i_1, y^i_2 \in \hat{P}^i(\bar{x})$, let $\lambda_1 \geq 0, \lambda_2 \geq 0$, such that $\lambda_1 + \lambda_2 = 1$, we show that $\lambda_1 y^i_1 + \lambda_2 y^i_2 \in \hat{P}^i(\bar{x})$. Then $y^i_k = \bar{x}^i + t_k(x^i_k - \bar{x}^i)$ for some $t_k \in [0,1]$ and some $x^i_k \in P^i(\bar{x})$ ($k = 1, 2$). One has

$$\lambda_1 y^i_1 + \lambda_2 y^i_2 = \bar{x}^i + (\lambda_1 t_1 + \lambda_2 t_2)(x^i - \bar{x}^i),$$

where $x^i := (\lambda_1 t_1 x^i_1 + \lambda_2 t_2 x^i_2)/((\lambda_1 t_1 + \lambda_2 t_2) \in P^i(\bar{x})$ (since $P^i(\bar{x})$ is convex, by Assumption (C)) and $\lambda_1 t_1 + \lambda_2 t_2 \in [0,1]$. Hence $\lambda_1 y^i_1 + \lambda_2 y^i_2 \in \hat{P}^i(\bar{x})$.

**Part (iii).** Let $y^i \in \hat{P}^i(\bar{x})$ and let $(x')^i \in X^i, (x')^i \neq y^i$. From the definition of $\hat{P}^i$, $y^i = \bar{x}^i + t(x^i - \bar{x}^i)$ for some $x^i \in P^i(\bar{x})$ and some $t \in [0,1]$. Suppose first that $x^i = (x')^i$, then $y^i \in [\bar{x}^i, x^i] \subset \hat{P}^i(\bar{x})$. Consequently, $[(x')^i, y^i \cap \hat{P}^i(\bar{x}) \neq \emptyset$. Suppose now that $x^i \neq (x')^i$; since $P^i$ satisfies Assumption (C.iii), there exists $\lambda \in [0,1]$ such that $x^i(\lambda) = (x')^i + \lambda(x^i - (x')^i) \in P^i(\bar{x})$. We let

$$z := [\lambda(1-t)\bar{x}^i + t(1-\lambda)(x')^i + t\lambda x^i]/\alpha \text{ with } \alpha := t + \lambda(1-t),$$

and we check that $z = [\lambda(1-t)\bar{x}^i + t\lambda x^i]/\alpha \in [\bar{x}^i, x^i(\lambda)]$, with $x^i(\lambda) \in P^i(\bar{x})$, hence $z \in \hat{P}^i(\bar{x})$. Moreover, $z := [\lambda(1-t)\bar{x}^i + t(1-\lambda)(x')^i]/\alpha \in [(x')^i, y^i$]. Consequently, $[(x')^i, y^i \cap \hat{P}^i(\bar{x}) \neq \emptyset$, which ends the proof of (iii).

**Parts (iv), (v) and (vi).** They follow immediately by the definition of $\hat{P}^i$ and the properties satisfied by $P^i$ in (C).

### 4.2.2 Truncating the Economy

Given $q \in \mathbb{R}^3$ the set of admissible consumptions and income transfers, $K(q)$ is given by:

$$K(q) := \{(x, w) \in \prod_{i \in \mathcal{I}} X^i \times \prod_{i \in \mathcal{I}} W^i : \exists p \in B_L(0,1), \sum_{i \in \mathcal{I}} x^i = \sum_{i \in \mathcal{I}} w^i = 0\}.$$

**Lemma 4.3** $K(q)$ is bounded.

**Proof of Lemma 4.3:** Given $q \in \mathbb{R}^3$, for every $i \in \mathcal{I}$ define the following:

$$\hat{X}^i(q) := \left\{x^i \in X^i : \exists (x^i)_{j \neq i} \in \prod_{j \neq i} X^j, \exists w \in \prod_{i \in \mathcal{I}} W^i, (x, w) \in K(q)\right\}$$

and

$$\hat{W}^i(q) := \left\{w^i \in W^i : \exists (w^j)_{j \neq i} \in \prod_{j \neq i} W^j, \exists x \in \prod_{i \in \mathcal{I}} X^i, (x, w) \in K(q)\right\}.$$

We need to show that $\hat{X}^i(q)$ and $\hat{W}^i(q)$ are bounded. Since $\hat{X}^i$ is compact (by Assumption C (i)), clearly $\hat{X}^i(q)$ is bounded.
To show $\hat{W}^i(q)$ is bounded, let $w^i \in \hat{W}^i(q)$. Since 

$$(x^i, w^i) \in \{(x, w) \in X^i \times W(q)Z^i \mid p \square (x - e^i) \leq w\}$$

and $(x^i, p) \in \hat{X}^i(q) \times B_L(0, 1)$, a compact set from above, $\exists \alpha^i \in \mathbb{R}^D$, such that

$$\alpha^i \leq p \square (x^i - e^i) \leq w^i$$

Using the fact that $\sum_{i \in I} w^i = 0$ we also have

$$w^i = - \sum_{j \neq i} w^j \leq - \sum_{j \neq i} \alpha^j,$$

Thus $\hat{W}^i(q)$ is bounded for every $i \in I$. $\Box$

We now define the "truncated economy" as follows.

Since $\hat{X}^i(q)$ and $\hat{W}^i(q)$ are bounded subsets of $\mathbb{R}^p$ and $\mathbb{R}^D$, respectively (by Lemma 4.3), there exists a real number $r > 0$ such that, for every agent $i \in I$, $\hat{X}^i(q) \subset \text{int} B_L(0, r)$ and $\hat{W}^i(q) \subset \text{int} B_L(0, r)$. The truncated economy $(\hat{\mathcal{E}}_r, \mathcal{F}_r)$ is the collection

$$(\hat{\mathcal{E}}_r, \mathcal{F}_r) = [\mathcal{D}, \mathcal{H}, \mathcal{I}, (X^i_r, \hat{\mathcal{P}}^i_r, e^i)_{i \in I}; \mathcal{J}, (\xi(j), V^j)_{j \in b J}, (Z^i_r)_{i \in I}],$$

where,

$$X^i_r = X^i \cap B_L(0, r), Z^i_r = \{z \in Z^i \mid W(q)z \in B_D(0, r)\} \text{ and } \hat{\mathcal{P}}^i_r(x) = \hat{\mathcal{P}}^i(x) \cap \text{int} B_L(0, r).$$

The existence of a weak equilibrium of $(\hat{\mathcal{E}}_r, \mathcal{F}_r)$ is then a consequence of Section 4.1, that is, Theorem 3.2 with the additional Assumption (K). We just have to check that Assumption (K) and all the assumptions of Theorem 3.2 are satisfied by $(\hat{\mathcal{E}}_r, \mathcal{F}_r)$. In view of Proposition 4.1, this is clearly the case for all the assumptions but the Survival Assumption ($C.vi$) that is proved via a standard argument (that we recall hereafter).

Indeed we first notice that $(e^i, 0)_{i \in I}$ belongs to $K(q)$, hence, for every $i \in I$, $e^i \in \hat{X}^i(q) \subset \text{int} B_L(0, r)$. Recalling that $e^i \in \text{int} X^i$ (from the Survival Assumption), we deduce that $e^i \in \text{int} X^i \cap \text{int} B_L(0, r) \subset \text{int} [X^i \cap B_L(0, r)] = \text{int} X^i_r$.

**Proposition 4.2** Given $\bar{q} \in \mathbb{R}^J$, if $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a weak equilibrium of $(\hat{\mathcal{E}}_r, \mathcal{F}_r)$ then it is also a weak equilibrium of $(\mathcal{E}, \mathcal{F})$.

**Proof of proposition 4.2.** Let $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ be a weak equilibrium of the economy $(\hat{\mathcal{E}}_r, \mathcal{F}_r)$. In view of the definition of a weak equilibrium, to prove that it is also a weak equilibrium of $(\mathcal{E}, \mathcal{F})$ we only have to check, for every $i \in I$, $[\mathcal{P}^i(\bar{x}) \times Z^i] \cap B^i(\bar{p}, \bar{q}) = \emptyset$, where $B^i(\bar{p}, \bar{q})$ denotes the budget set of agent $i$ in the economy $(\mathcal{E}, \mathcal{F})$. 30
Assume, on the contrary, that, for some \( i \in I \) the set \( [P^i(\bar{x}) \times Z]^i \cap B^i(\bar{p}, \bar{q}) \) is nonempty, hence contains a couple \((x^i, z^i)\). Clearly the allocation \((\bar{x}, W(\bar{q})\bar{z})\) belongs to the set \( K(\bar{q}) \), hence for every \( i \in I \), \( \bar{x}^i \in \tilde{X}^i(\bar{q}) \subset \text{int}B_L(0, r) \) and \( \bar{w}^i = W(\bar{q})\bar{z}^i \in \tilde{W}^i(\bar{q}) \subset \text{int}B_D(0, r) \). Thus, for \( t \in [0, 1] \) sufficiently small, \( x^i(t) := \bar{x}^i + t(x^i - \bar{x}^i) \in \text{int}B_L(0, r) \) and \( w^i(t) := \bar{w}^i + t(w^i - \bar{w}^i) \in \text{int}B_D(0, r) \). Clearly \((x^i(t), w^i(t))\) is such that \( w^i(t) = W(\bar{q})z^i(t) \) where \( z^i(t) = (\bar{z}^i + t(z^i - \bar{z}^i)) \in Z^i \) and \((x^i(t), z^i(t))\) belongs to the budget set \( B^i(\bar{p}, \bar{q}) \) of agent \( i \) (for the economy \((E, F)\)). From the definition of \( \tilde{P}^i \), we deduce that \( x^i(t) \in \tilde{P}^i(\bar{x}) \) (since from above \( x^i(t) := \bar{x}^i + t(x^i - \bar{x}^i) \) and \( x^i \in P^i(\bar{x}) \)), hence \( x^i(t) \in \tilde{P}^i(\bar{x}) \cap \text{int}B_L(0, r) \). We have thus shown that, for \( t \in [0, 1] \) small enough, \((x^i(t), z^i(t))\) belongs also to the budget set \( B^i(\bar{p}, \bar{q}) \) of agent \( i \) (in the economy \((\tilde{E}_r, \tilde{F}_r)\)). From the definition of \( \tilde{P}^i \), we deduce that \((x^i(t), z^i(t))\) belongs also to the budget set \( B^i(\bar{p}, \bar{q}) \) of agent \( i \) (in the economy \((\tilde{E}_r, \tilde{F}_r)\)). From the definition of \( \tilde{P}^i \), we deduce that \((x^i(t), z^i(t))\) belongs also to the budget set \( B^i(\bar{p}, \bar{q}) \) of agent \( i \) (in the economy \((\tilde{E}_r, \tilde{F}_r)\)). The contradiction follows. \( \square \)
References


