Affiliation, Equilibrium Existence and the Revenue Ranking of Auctions*

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Abstract

We consider private value auctions where bidders' types are dependent, a case usually treated by assuming affiliation. As any scientific assumption, affiliation has limitations and it is important to know them. We show that affiliation is topologically and measure-theoretically restrictive. The economic cases where affiliation is justified do not correspond to the intuition usually given for introducing affiliation. Affiliation's implications do not generalize to other definitions of positive dependence and may be false in general. Nevertheless, we show that some of these implications are true in weaker senses. Also, there are cases where affiliation can be well justified and used in theoretical models. However, since these cases do not cover all economically relevant cases, there is space for a more general approach to dependence in auctions.

We propose a new approach that allow both theoretical and numerical characterization of auctions. We treat mainly symmetric auctions, but the approach can be extended to asymmetric auctions with dependence. New results about equilibrium existence and revenue ranking of auctions are provided.

JEL Classification Numbers: C62, C72, D44, D82.

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1 Introduction

Private information is a central theme in modern economics. It is usually introduced in the economic models through (privately known) random variables. It is mathematically

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convenient to assume that such random variables are independent, but this assumption is unrealistic.

The importance of understanding dependence in auction theory was early recognized (see Wilson 1969 and 1977). A remarkable contribution to this problem was made through an insight of Milgrom and Weber (1982a), who introduced the concept of affiliation in auction theory.[1]

Affiliation is a generalization of independence — see the definition in section 3 — that was explained through the appealing notion that bidders' assessments of the object are positively dependent: "Roughly, this [affiliation] means that a high value of one bidder's estimate makes high values of the others' estimates more likely." (Milgrom and Weber (1982a), p. 1096.)

Among the many results implied by affiliation and obtained by Milgrom and Weber (1982a), we underline the following: (i) affiliation implies the existence of a symmetric, increasing, pure strategy equilibrium for first price auctions;[2] (ii) under affiliation, the English and the second price auction have higher expected revenue than the first price auction.[3]

In the face of these results, it is possible to cite at least three reasons for the profound influence of the aforementioned paper in auction theory: (i) its theoretical depth and elegance; (ii) the plausibility of the hypothesis of affiliation, as explained by a clear economic intuition; (iii) the fact that it implies that English auctions are better for sellers, which is a good explanation for the fact that English auctions are more common than first price auctions.

Nevertheless, since we were originally interested in the general problem of dependence of types, it is important to have an assessment of how strong as assumption affiliation is and how robust its implications are.

In section 3 we show that affiliation is quite restrictive in two senses. It is restrictive in a *topological* sense: the set of no affiliated probability density functions (p.d.f.'s) is open and dense in the set of continuous p.d.f.'s. It is also restrictive in a *measure-theoretic* sense: if μ is a probability measure over the set of joint probabilistic density functions (p.d.f.'s), and if μ satisfies some weak conditions, then then μ puts zero measure in the set of affiliated p.d.f.'s.

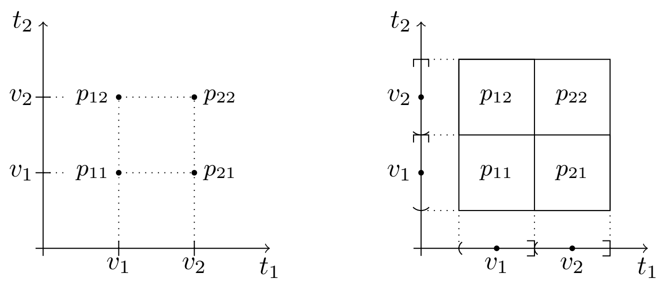
[1]In two previous papers, Paul Milgrom presented results that use a particular version of the same concept, under the name "monotone likelihood ratio property" (MLRP): Milgrom (1981a, 1981b). It is also likely that Wilson (1969 and 1977) has influenced the development of the affiliation idea. Nevertheless, the concept is fully developed and the term affiliation first appears in Milgrom and Weber (1982a). See also Milgrom and Weber (1982b). When there is a density function, the property had been previously studied by statisticians under different names. Lehmann (1966) calls it Positive Likelihood Ratio Dependence (PLRD), Karlin (1968) calls it Total Positivity of order 2 (TP-2) for the case of two variables or Multivariate Total Positivity of Order 2 (MTP-2) for the multivariate case. Others use the name Monotone Likelihood Ratio Property (for the bivariate case). Nevertheless, it seems that Milgrom and Weber (1982a) were the first to generalize the definition from the case where there is a density function for the general case, where is assumed only the existence of a joint distribution.

[2]They also proved the existence of equilibrium for second price auctions with interdependent values. In our set-up (private values), the second price auction always has an equilibrium in weakly dominant pure strategies, which simply consists of bidding the private value. Although equilibria in mixed strategies always exist (Jackson and Swinkels 2005), first price auctions may fail to possess a pure strategy equilibrium when types are dependent.

[3]For private value auctions, which we consider in this paper, English and second price auctions are equivalent — see Milgrom and Weber (1982a). Thus, we restrict our analysis to the latter.

From this, we reexamine the intuition used to introduce affiliation: the positive dependence intuition given above. In Statistics, there are many definitions of positive dependence, and affiliation is just one of the most restrictive. We show that there are some economic models where affiliation is ensured, although they do not cover all the economically relevant cases. We also show that the main implications of affiliation — equilibrium existence and the revenue ranking of auctions — do not extend for other (still restrictive) definitions of positive dependence. From this, it seems important to reconsider the problems of equilibrium existence and revenue ranking of auctions with dependent types.

For solving the problem of symmetric increasing pure strategy equilibrium existence, we restrict the set of distributions considered. The idea is so simple that it can be explained graphically (see Figure 1).



(a) $p_{ij} = \Pr(t_1 = v_i, t_2 = v_j)$ (b) $p_{ij} = f(v_i, v_j)$

Figure 1: Discrete values, such as in (a), capture the relevant economic possibilities in a private value model, but preclude the use of calculus. We use continuous variables, but consider only simple density functions (constant in squares), such as in (b).

Let us expand the above explanation. Consider the setting of symmetric private value auctions with two risk neutral players, but general dependence of types. Since we are analyzing auctions of single objects, it would be sufficient to consider the case where bidders' types are distributed according to a finite number of values (the values can be specified only up to cents and are obviously bounded). Nevertheless, to work with discrete values precludes us from using the convenient tools of differential calculus, which allow, for instance, a complete characterization of equilibrium strategies. Maintaining the advantage of continuum variables, but without requiring unnecessary richness in the set of distributions, we focus on the set of densities which are constant in some squares around fixed values. This imposes no economic restriction on the cases considered, but allows a complete characterization of symmetric increasing pure strategy equilibrium (PSE) existence (see subsection 4.1).

It is easy to see that, as we take arbitrarily small squares, we can approximate any p.d.f. (including non-continuous ones). Thus, even if the reader insists on mathematical generality, that is, to include other distributions, our results are still meaningful because they cover a dense set.

For this set of simple p.d.f.'s, we are able to provide an algorithm, implementable by a computer, that completely characterizes whether or not a pure strategy equilibrium exists. Theoretical results are also available.

The results in section 4 reveal that the set of affiliated distribution is small even in the set of distributions with a PSE, sharpening the results of restrictiveness of affiliation. On the other hand, the proportion of p.d.f.'s which have PSE is also small. We offer a mathematical proof of this fact. This suggests that most of the cases only have equilibria in mixed strategies.

From this, we consider the characterization of the revenue ranking of auctions. The standard approach in the literature (see Milgrom and Weber 1982 for affiliated distributions and Maskin and Riley 2000 for asymmetrical independent distributions) is to give conditions on the set of distributions that imply such or such ranking. Unfortunately, the conditions are usually very restrictive.

Numerical simulations, made possible by the results of sections 4 and 5, suggest that a complete characterization is very difficult. Even p.d.f.'s with positive dependence may often present a revenue ranking contrary to that implied by affiliation.

From this, we see two options for approaching the problem of revenue ranking. One is to obtain (experimental or empirical) information related to the specific situation and restrict the set of p.d.f.'s to analyze. Then, with this restriction, use our model to run simulations and determine what auction format gives higher expected revenue in the specific environment. Although this "engineering"-type approach was recently proposed by Roth (2002) as a tool for economists, it will be useful to also have a general, theoretical model. This is the other part of the proposal.

We construct a "natural" measure over our proposed set of p.d.f.'s — this is natural in the sense that it comes from the limit of Lebesgue measure over finite-dimensional sets and also has some (partial) characteristics of Lebesgue measure. Using this measure, we obtain an "expected value" of the expected revenue difference of the second-price and the first-price auction.

We illustrate this theoretical (and specification-free) method in section 5. This method suggests that the English auction gives higher revenue than the first price auction "on average". Surprisingly, this conclusion seems to hold in general, not only for positive dependent distributions.[4]

Thus, our paper makes the following contributions: it illustrates the restrictiveness of affiliation; proposes a convenient and sufficiently general set of distributions; offers a method to numerically test equilibrium existence and the revenue ranking of auctions for non-affiliated distributions; and shows that the revenue ranking implied by affiliation is valid, in a weak sense, for a bigger set of distributions.

The paper is organized as follows. Section 2 gives a brief exposition of the standard auction model. Section 3 compares affiliation and other definitions of positive dependence and shows that affiliation is a restrictive condition. Section 4 presents the equilibrium existence results. Section 5 describes the proposed methods for approaching the problem of revenue ranking in auctions. Section 6 contains a comparison with related literature and concluding remarks. The more important and short proofs are given in an appendix, while lengthy constructions are presented in a separate supplement to this paper.

[4]The reader should not be confused: the conclusion is in a weak sense: just "on average". For specific distributions, as we said, the rank can be in any direction.

2 Model and definitions

Our model and notations are standard. There are n bidders, $i = 1, ..., n$. Bidder i receives private information $t_i \in \left[\underline{t}, \overline{t}\right]$ which is the value of the object for himself. As usual we will denote the profile of types $t = (t_1, ..., t_n) \in \left[\underline{t}, \overline{t}\right]^n$ by $t = (t_i, t_{-i})$. The values are distributed according to a p.d.f. $f : \left[\underline{t}, \overline{t}\right]^n \rightarrow \mathbb{R}_+$ which is symmetric, that is, if $\pi : \{1, ..., n\} \rightarrow \{1, ..., n\}$ is a permutation, $f(t_1, ..., t_n) = f\left(t_{\pi(1)}, ..., t_{\pi(n)}\right)$.[5] Let $\overline{f}(x) = \int f(x, t_{-i})\, dt_{-i}$ be a marginal of f. Our main interest is the case where f *is not* the product of its marginals, that is, the case where the types are dependent. We denote by $f(t_{-i}|t_i)$ the conditional density $f(t_i, t_{-i}) / \overline{f}(t_i)$. After knowing his value, bidder i places a bid $b_i \in \mathbb{R}_+$. He receives the object if $b_i > \max_{j \neq i} b_j$. We consider both first and second price auctions. As Milgrom and Weber (1982a) argue, second price and English auctions are equivalent in the case of private values, as we assume here. In a first price auction, if $b_i > \max_{j \neq i} b_j$, bidder i's utility is $u(t_i - b_i)$ and is $u(0) = 0$ if $b_i < \max_{j \neq i} b_j$. In a second price auction, bidder i's utility is $u(t_i - \max_{j \neq i} b_j)$ if $b_i > \max_{j \neq i} b_j$ and $u(0) = 0$ if $b_i < \max_{j \neq i} b_j$. For both auctions, ties are randomly broken.

By reparametrization, we may assume, without loss of generality, $\left[\underline{t}, \overline{t}\right] = [0, 1]$. It is also useful to assume $n = 2$, but this is not needed for most of the results, although some (especially in sections 4 and 5) may require non-trivial adaptations for $n > 2$. For most of the paper, we assume risk neutrality, that is, $u(x) = x$. Thus, unless otherwise stated, the results will be presented under the following set-up:

BASIC SETUP: *There are $n = 2$ risk neutrals bidders, that is, $u(x) = x$, with private values distributed according to a symmetric density function $f : [0, 1]^2 \rightarrow \mathbb{R}_+$.*

A pure strategy is a function $b : [0, 1] \rightarrow \mathbb{R}_+$, which specifies the bid $b(t_i)$ for each type t_i. The interim payoff of bidder i, who bids β when his opponents follow $b : [0, 1] \rightarrow \mathbb{R}_+$ is given by

$$\Pi_i(t_i, \beta, b) = u(t_i - \beta) F\left(b^{-1}(\beta) \mid t_i\right) = u(t_i - \beta) \int_{\underline{t}}^{b^{-1}(\beta)} f(t_{-i} \mid t_i)\, dt_{-i},$$

if it is a first price auction and

$$\Pi_i(t_i, \beta, b) = \int_{\underline{t}}^{b^{-1}(\beta)} u(t_i - b(t_{-i}))\, f(t_{-i} \mid t_i)\, dt_{-i},$$

if it is a second price auction.

We focus attention on symmetric increasing pure strategy equilibrium (PSE), which is defined as a function $b : [0, 1] \rightarrow \mathbb{R}_+$ such that $\Pi_i(t_i, b(t_i), b) \geq \Pi_i(t_i, \beta, b)$ for all β and t_i. The usual definition requires this inequality to be true only for almost all t_i. This stronger definition creates no problem and makes some statements simpler, as those about the differentiability and continuity of the equilibrium bidding function

[5] For the reader familiar with Mertens and Zamir (1986)'s construction of universal type spaces: we make the usual assumption in auction theory that the model is "closed" at the first level, that is, all higher level beliefs are consistently given by (and collapse to) f.

(otherwise, such properties should be always qualified by the expression "almost everywhere"). Finally, under our assumptions, the second price auction always has a PSE in a weakly dominant strategy, which is $b(t_i) = t_i$.

3 Affiliation and Positive Dependence

A key aspect of most auctions is that bidders' values are private information. These private pieces of information are usually modeled as random variables. Although it is mathematically easier to assume that such random variables are independent, this is unrealistic.[6] Indeed, many real world institutions can act as correlation devices: culture, education, common sources of information, evolution, etc. Thus, a deeper understanding of auctions requires dealing with dependence of private information.

The introduction of affiliation in auction theory was a milestone in this enterprise. Milgrom and Weber (1982a) borrowed a statistical concept (multivariate total positivity, MTP) and applied it to a general model of symmetric auctions.[7] In this fashion, they were able to prove many important results, including a revenue ranking that is different from the provisions of the revenue equivalence theorem (Vicrkey, 1961 and Myerson, 1981). The formal definition is as follows:

Definition 1 *The density function* $f : \left[\underline{t}, \bar{t}\right]^n \to \mathbb{R}_+$ *is affiliated if* $f(t) f(t') \leqslant f(t \wedge t') f(t \vee t')$*, where* $t \wedge t' = (\min\{t_1, t_1'\}, ..., \min\{t_n, t_n'\})$ *and* $t \vee t' = (\max\{t_1, t_1'\}, ..., \max\{t_n, t_n'\})$.[8]

For a quarter of a century, auction theorists used affiliation's properties to derive important conclusions. Affiliation's monotonicity properties (see Theorem 5 of Milgrom and Weber, 1982a) combine well with natural properties of auctions, simplifying the analysis and allowing useful predictions. In sum, affiliation provided foundation for a successful theory, as auction theory is considered (see e.g. Maskin, 2004).

However, as any scientific acchievement, affiliation has limitations. It is important to know what and how relevant such limitations are. The purpose of this section is to offer an assessment of these aspects. In subsection 3.1, we show that the set of affiliated distributions is a small subset or, in other words, affiliation is a restrictive assumption. Although this observation is interesting, it is more important to know whether the assumption is reasonable in economically relevant situations and how robust its implications are. For this, in subsection 3.2 we review the economic intuition used to describe affiliation and, in subsection 3.3, some cases where affiliation holds. We review some of the affiliation's implications in subsection 3.4. A summary of the findings is presented in subsection 3.5.

[6] For some problems, as those considered in mechanism design, there is also an *economic* justification for assuming independence. As shown by Crémer and McLean (1987), dependence of types can allow full extraction of the bidders' surplus. This is not important for the problems that we are considering here, where the mechanisms are fixed (first and second price auctions).

[7] Milgrom and Weber (1982) generalize the definition of MTP to the case of distributions without density functions. It is possible, but unclear, that this generalization is also an original contribution of them.

[8] In Statistics and Probability, this is called FKG inequality.

3.1 The set of affiliated densities is small

In this subsection, we show that affiliation is a restrictive assumption, that is, the set of affiliated densities is small in the set of all densities. There are two ways to characterize a set as small: topological and measure-theoretic. We consider both in the sequel, beginning with the topological.

Let $\mathcal{C}$ denote the set of continuous density functions $f : [0,1]^2 \to \mathbb{R}_+$ and let $\mathcal{A}$ be the set of affiliated densities. For convenience and consistency with the notation in next sections, we are including in $\mathcal{A}$ all affiliated densities and not only the continuous one, which creates no problem. Endow $\mathcal{C}$ with the topology of the uniform convergence, that is, the topology defined by the norm of the sup:

$$\|f\| = \sup_{x \in [0,1]^2} |f(x)| .$$

The following theorem shows that the set of continuous affiliated densities is small in the topological sense.

Theorem 2 *The set of continuous affiliated density function $\mathcal{C} \cap \mathcal{A}$ is meager. More precisely, the set $\mathcal{C} \backslash \mathcal{A}$ is open and dense in $\mathcal{C}$.*
Proof. *See the appendix.* ■

In fact, the theorem says more than the set of continuous affiliated density functions is a meager set. A meager set (or set of first category) is the union of countably many nowhere dense sets, which are sets whose closure has empty interior. $\mathcal{C} \cap \mathcal{A}$ is itself a nowhere dense set, by the second claim in the theorem.

The proof of this theorem is given in the appendix, but the main idea is simple. To prove that $\mathcal{C} \backslash \mathcal{A}$ is open, we take a p.d.f. $f \in \mathcal{C} \backslash \mathcal{A}$ which does not satisfy the affiliated inequality for some points $t, t' \in [0,1]^2$, that is, $f(t) f(t') > f(t \wedge t') f(t \vee t') + \eta$, for some $\eta > 0$. By using such η, we can show that for a function g sufficiently close to f, the above inequality is still valid, that is, $g(t) g(t') > g(t \wedge t') g(t \vee t')$ and, thus, is not affiliated. To prove that $\mathcal{C} \backslash \mathcal{A}$ is dense, we choose a small neighborhood V of a point $\hat{t} \in [0,1]^2$, such that for all $t \in V$, $f(t)$ is sufficiently close to $f\left(\hat{t}\right)$ — this can be done because f is continuous. Then, we perturb the function in this neighborhood to get a failure of the affiliation inequality.

Maybe more instructive than the proof is to understand why the result is true: simply, affiliation requires an inequality to be satisfied everywhere (or almost everywhere). This is a strong requirement and it is the source of its restrictiveness.

Affiliation is also restrictive in the measure-theoretic sense, that is, in an informal way, it is of "zero measure". Obviously, we need to be careful with the formalization of this, since we are now dealing with measures over infinite-dimensional sets (the set of distributions or densities). As is well known, there are no "natural" measures for infinite dimension sets, that is, measures with all of the properties of the Lebesgue measure — see Yasamaki (1985), Theorem 5.3, p. 139.

Thus, before we formalize our results, we informally explain what we mean by "measure-theoretic". Let $\mathcal{D}$ be the set of probabilistic density functions (p.d.f.'s) $f : [0,1]^2 \to \mathbb{R}_+$ and assume that there is a measure μ over it. We define below a sequence

$\mathcal{D}^k$ of finite-dimensional subspaces of $\mathcal{D}$ and take the measures μ^k over $\mathcal{D}^k$ induced by the projection of $\mathcal{D}$ over $\mathcal{D}^k$. The result is as follows: if μ^k is absolutely continuous with respect to the Lebesgue measure λ^k over $\mathcal{D}^k$ — as seems reasonable — then μ puts zero measure on the set $\mathcal{A}$ of affiliated p.d.f.'s.

Remark. There is an alternative method of characterizing smallness in the measure-theoretic sense: to show that the set is *shy*, as defined by Anderson and Zame (2001), generalizing a definition of Christensen (1974) and Hunt, Sauer and Yorke (1992). We discuss it in the supplement to this paper.

Now, we formalize our method. Endow $\mathcal{D}$ with the L^1-norm, that is, $\|f\|_1 = \int |f(t)|\, dt$. When there is no peril of confusion with the sup norm previously defined, we write $\|f\|$ for $\|f\|_1$.

For $k \geq 2$, define the transformation $T^k : \mathcal{D} \to \mathcal{D}$ by

$$T^k(f)(x,y) = k^2 \int_{\frac{p-1}{k}}^{\frac{p}{k}} \int_{\frac{m-1}{k}}^{\frac{m}{k}} f(\alpha,\beta)\, d\alpha d\beta,$$

whenever $(x,y) \in \left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$, for $m, p \in \{1, 2, ..., k\}$. Observe that $T^k(f)$ is constant over each square $\left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$. Let $\mathcal{D}^k$ be the image of $\mathcal{D}$ by T^k, that is, $\mathcal{D}^k \equiv T^k(\mathcal{D})$. Thus, T^k is a projection.

Observe that $\mathcal{D}^k$ is a finite dimensional set. In fact, a density function $f \in \mathcal{D}^k$ can be described by a matrix $A = (a_{ij})_{k\times k}$, as follows:

$$f(x,y) = a_{mp} \text{ if } (x,y) \in \left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right], \tag{1}$$

for $m, p \in \{1, 2, ..., k\}$. The definition of f at the zero measure set of points $\{(x,y) = \left(\frac{m}{k}, \frac{p}{k}\right) : m = 0 \text{ or } p = 0\}$ is arbitrary.

The following result is important to our method:

Proposition 3 *f is affiliated if and only if for all k, $T^k(f)$ also is. In mathematical notation: $f \in \mathcal{A} \Leftrightarrow T^k(f) \in \mathcal{A}$, $\forall k \in \mathbb{N}$, or yet: $\mathcal{A} = \cap_{k\in\mathbb{N}} T^{-k}\left(\mathcal{A} \cap \mathcal{D}^k\right)$.*
Proof. *See the supplement to this paper.* ■

The set of affiliated distributions $\mathcal{A}$ is the countable intersection of the sets $T^{-k}\left(\mathcal{A} \cap \mathcal{D}^k\right)$, and these sets themselves are small. $T^{-k}\left(\mathcal{A} \cap \mathcal{D}^k\right)$ is small in $\mathcal{D}$ because $\mathcal{A} \cap \mathcal{D}^k$ is small in $\mathcal{D}^k$ (by definition, T^k is surjective). In fact, we have the following:

Proposition 4 *If λ^k denotes the Lebesgue measure over $\mathcal{D}^k$, then $\lambda^k\left(\mathcal{A} \cap \mathcal{D}^k\right) \downarrow 0$ as $k \to \infty$.*
Proof. *See the supplement to this paper.* ■

The convergence is extremely fast, as shown in the following table, obtained by numerical simulations, with 10^7 cases (see the supplement to this paper for the description of the numerical simulation method and other results).

	$k=3$	$k=4$	$k=5$	$k=6$
$\lambda^k\left(\mathcal{A}\cap\mathcal{D}^k\right)$	1.1%	~0.01%	$\sim 10^{-6}$	$<10^{-7}$

Table 1 - Proportion of affiliated distribution
in the sets $\mathcal{D}^k$.

Now, define the measure μ^k over $\mathcal{D}^k$ as follows: if $E \subset \mathcal{D}^k$ is a measurable subset, put $\mu^k(E)= \mu\left(T^{-k}(E)\right)$. Now, it is easy to obtain the main result of this subsection:

Theorem 5 *If $\mu^k \leq M\lambda^k$ for some $M > 0$ then, $\mu(\mathcal{A}) = 0$.*[9]
Proof. *By Proposition 3, $\mathcal{A} \subset T^{-k}\left(\mathcal{A}\cap\mathcal{D}^k\right)$ for every k. Thus,*

$$\mu(\mathcal{A}) \leq \mu\left(T^{-k}\left(\mathcal{A}\cap\mathcal{D}^k\right)\right) = \mu^k\left(\mathcal{A}\cap\mathcal{D}^k\right) \leq M\lambda^k\left(\mathcal{A}\cap\mathcal{D}^k\right).$$

Since $\lambda^k\left(\mathcal{A}\cap\mathcal{D}^k\right) \downarrow 0$ as $k \to \infty$, by Proposition 4, we have the conclusion. ■

As the reader may note from the above proof, it is possible to change the condition $\mu^k \leq M\lambda^k$ for some $M > 0$ by $\mu^k \leq M^k\lambda^k$ for a sequence M^k, as long as M^k does not go to infinity as fast as $\lambda^k\left(\mathcal{A}\cap\mathcal{D}^k\right)$ goes to zero. Since the convergence $\lambda^k\left(\mathcal{A}\cap\mathcal{D}^k\right) \downarrow 0$ is extremely fast, as we noted above, this assumption seems mild.

It is useful to observe that Theorem 5 is not empty, that is, there are many measures over $\mathcal{D}$ that satisfy it. A way to see this is to recall that a measure over $\mathcal{D}$ can be constructed from the measures over the finite-dimensional sets $\mathcal{D}^k$ by appealing to the Kolmogorov Extension Theorem (see Aliprantis and Border 1991, p. 491). The interested reader will find more comments about this in the supplement to this paper.

Although the results presented in this section are new, they are not surprising for many auction specialists. It seems to be known that affiliation is restrictive, but this may be of less importance, if affiliation is valid in the economically relevant situations. Because of this, in the next subsection we discuss the cases where affiliation is supposed to hold.

3.2 The intuition for affiliation

Affiliation was introduced through the positive dependence intuition: "a high value of one bidder's estimate makes high values of the others' estimates more likely", Milgrom and Weber (1982a), p. 1096. This intuition is very appealing, since positive

[9]The reader may note that the assumption is slightly stronger than absolute continuity of μ^k with respect to λ^k. In fact, absolute continuity requires only that $\lambda^k(A) = 0$ implies $\mu^k(A) = 0$. Nevertheless, by the Radon-Nikodym Theorem, absolute continuity implies the existence of a measurable function m^k such that $\mu^k(A) = \int_A m^k d\lambda^k$. Thus, the above assumption is really requiring this function m^k to be bounded: $m^k \leq M$. As we discuss in the paragraph after the Theorem, this bound does not need to be uniform in k.

dependence is likely the most common kind of dependence in the real world. In fact, many authors introduce affiliation through this intuition or some of its variances.

It is easy to see that affiliation captures this intuition, as we illustrate in Figure 2, below. Affiliation requires that the product of the weights at points (x', y') and (x, y) (where both values are high or both are low) be greater than the product of weights at (x, y') and (x', y) (where they are high and low, alternatively). In other words, the distribution puts more weight in the points in the diagonal than outside it.

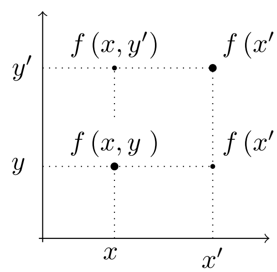


Figure 2: The p.d.f. f is affiliated if $x \leq x'$ and $y \leq y'$ imply $f(x, y') f(x', y) \leq f(x', y') f(x, y)$.

However, as long as we are interested in *positive dependence*, as the given intuition suggests, affiliation is not the only definition available. In the statistical literature many concepts were proposed to correspond to the notion of positive dependence. For simplicity, let us consider only the bivariate case, and assume that the two real random variables X and Y have joint distribution F and strictly positive density function f. The following concepts are formalizations of the notion of positive dependence:[10]

Property I - X and Y are positively correlated (PC) if $cov(X, Y) \geqslant 0$.

Property II - X and Y are said to be positively quadrant dependent (PQD) if $cov(g(X), h(Y)) \geqslant 0$, for all non-decreasing functions g and h.

Property III - The real random variables X and Y are said to be associated (As) if $cov(g(X, Y), h(X, Y)) \geqslant 0$, for all non-decreasing functions g and h.

Property IV - Y is said to be left-tail decreasing in X (denoted LTD($Y|X$)) if $\Pr[Y \leqslant y | X \leqslant x]$ is non-increasing in x for all y. X and Y satisfy property IV if LTD($Y|X$) and LTD($X|Y$).

Property V - Y is said to be positively regression dependent on X (denoted PRD($Y|X$)) if $\Pr[Y \leqslant y | X = x] = F(y|x)$ is non-increasing in x for all y. X and Y satisfy property V if PRD($Y|X$) and PRD($X|Y$).

[10]Most of the concepts can be properly generalized to multivariate distributions. See, e.g., Lehmann (1966) and Esary, Proschan and Walkup (1967). The hypothesis of strictly positive density function is made only for simplicity.

Property VI - Y is said to be Inverse Hazard Rate Decreasing in X (denoted IHRD($Y|X$)) if $\frac{F(y|x)}{f(y|x)}$ is non-increasing in x for all y, where $f(y|x)$ is the p.d.f. of Y conditional to X. X and Y satisfy property VI if IHRD($Y|X$) and IHRD($X|Y$).

We have the following:

Theorem 6 *Let affiliation be Property VII. Then, the above properties are successively stronger, that is,*

$$(VII) \Rightarrow (VI) \Rightarrow (V) \Rightarrow (IV) \Rightarrow (III) \Rightarrow (II) \Rightarrow (I)$$

and all implications are strict.
Proof. *See the appendix.*[11] ■

For this theorem, we used only seven concepts for simplicity. Yanagimoto (1972) defines more than thirty concepts of positive dependence and, again, affiliation is the most restrictive of all, but one.

The main contribution of this subsection is not the mathematical result presented as Theorem 6, but the observation that: 1) positive dependence was our primary target in the study of dependence in auctions; 2) affiliation *is not* positive dependence but just one among many possible definitions; and 3) affiliation is one of the most restrictive among those definitions.

This observation is important for an assessment of the assumption. If we believe that positive dependence corresponds to the set of economically relevant cases, then affiliation may not be the correct assumption or, in other words, the received intuition may be misleading. Accepting the intuition, we may believe that we are covering exactly the important cases, when we are not. The contribution here is to warn of this potential gap.

Of course, we may think that the positive dependence cases are *not*, in fact, the economically relevant ones. Instead, maybe the economically relevant cases are exactly those where affiliation holds. Thus, we need to consider more precisely these cases. This is the subject of the next subsection.

3.3 Economic models where affiliation is valid

There are meaningful economic models where affiliation is ensured. The first, trivial example is that of independence. Although independence is a restrictive assumption, there are economic situations where it can be safely assumed. For instance, if the knowledge of one bidder's private value does not change the belief about the other

[11] Some implications of Theorem 6 are trivial and most of them were previously established. Our contribution regards Property VI, that we use later to prove convenient generalizations of equilibrium existence and revenue rank results. We prove that Property VI is strictly weaker than affiliation and is sufficient for, but not equivalent to Property V.

bidders' values, then we can assume independence. In fact, it is conceivable that some situations follow this intuitive condition.

Important cases with actual dependence can be found in the conditional independence models. These models assume that the signals of the bidders are conditionally independent, given a variable v (which can be the intrinsic value of the object, see Wilson 1969, 1977). Some specialists seem to believe that symmetry (exchangeability, in de Finneti's terminology) implies conditional independence. Unfortunately, de Finneti's theorem is not valid for a finite number of random variables. (See Diaconis and Freedman (1980) for discussion and a partial generalization). Thus, assuming conditional independence *implies loss of generality*. This is not a problem *if* the case in study satisfies the intuition that bidders' assessment are conditionally independent (know one bidder's assessment does not change the believe about the other bidders' assessments), given some variable.

However, even if we are ready to assume a conditional independence model, some care is still necessary before getting affiliation. To see this, assume that the p.d.f. of the signals conditional to v, $f(t_1, ..., t_n|v)$, is C^2 (twice continuously differentiable) and has full support. It can be proven that the signals are affiliated if and only if

$$\frac{\partial^2 \log f(t_1, ..., t_n|v)}{\partial t_i \partial t_j} \geqslant 0,$$

and

$$\frac{\partial^2 \log f(t_1, ..., t_n|v)}{\partial t_i \partial v} \geqslant 0, \tag{2}$$

for all i, j.[12] It is important to note that conditional independence implies only that

$$\frac{\partial^2 \log f(t_1, ..., t_n|v)}{\partial t_i \partial t_j} = 0.$$

Thus, conditional independence is not sufficient for affiliation. To obtain the latter, one needs to assume (2) or that t_i and v are affiliated. In other words, to obtain affiliation from conditional independence, one has to assume affiliation itself. Thus, we still have the problem of justifying the last part (affiliation).

The fact that we are not able to find a justification in the general model of conditional independence does not imply that it does not exist, at least in special cases. In fact, there is a particular case of this model that can be fully justified. Assume that the signals t_i are a common value plus an individual error, that is, $t_i = v + \varepsilon_i$, where the ε_i are independent and identically distributed. Now, we *almost* have the result that the signals t_1, ..., t_n are affiliated: it is still necessary to assume an additional condition. Let g be the p.d.f. of the ε_i, $i = 1$, ..., n. Then, t_1, ..., t_n are affiliated if and only if g is a strongly unimodal function.[13,14]

[12] See Topkis (1978), p. 310.

[13] The term is borrowed from Lehmann (1959). A function is strongly unimodal if $\log g$ is concave. A proof of the affirmation can be found in Lehmann (1959), p. 509, or obtained directly from the previous discussion.

[14] Even if g is strongly unimodal, so that $t_1, ..., t_n$ are affiliated, it is not true in general that t_1, ..., t_n, ε_1, ..., ε_n, v are affiliated.

Another instance of justification of affiliation is when we have some reason to believe or accept that the bidders' values are distributed according to some specific distribution. If this distribution has the affiliation property, then the use of affiliation is justified by the reasons for adopting the distribution.

Yet, the inability of providing a good general economical justification for affiliation does not imply that afilliation is not useful. Even if an assumption is not valid in the real world, what is more important is its implications. As Friedman (1953) argues, the most important criterion for judging an assumption is whether the resulting theory "yields sufficiently accurate predictions"(p. 14). Because of this, we analyze affiliation's implications in the next subsection.

3.4 Implications of affiliation

Many results were proved using affiliation. They can be classified in two groups: facts that are already true for the independent case (affiliation allows a generalization) and predictions qualitatively different from the case of independence. In this subsection, we will focus in one example in each of these groups.

The first one is the (symmetric monotonic pure strategy) equilibrium (MPSE) existence for first price auctions, generalized from independence to affiliation. The second one is the revenue ranking of auctions: under affiliation, the English (and the second price) auction gives expected revenue at least as high as the first price auction ($R_2 \geq R_1$), in contrast with the case of independence, where the revenue equivalence theorem implies the equality of the expected revenues ($R_2 = R_1$).[15,16] Both implications were obtained by Milgrom and Weber (1982) and we choose them because of their importance. The purpose of this subsection is to verify whether these implications (MPSE existence and $R_2 \geq R_1$) are true in a more general setting.

Is MPSE existence true under other definitions of positive dependence (see subsection 3.2)? For private values auctions, it is not difficult to see that the same proof from Milgrom and Weber (1982a) can be used to prove equilibrium existence for Property VI. Indeed, the following property is sufficient:[17]

Property VI′ - The joint (symmetric) distribution of X and Y satisfy property VI′ if for all x, x' and y in $[0,1]$, $x \geq y \geq x'$ imply

$$\frac{F(y|x')}{f(y|x')} \geq \frac{F(y|y)}{f(y|y)} \geq \frac{F(y|x)}{f(y|x)}.$$

It is easy to see that Property VI implies Property VI′ (under symmetry and full support). Thus, the question becomes whether the generalization of MPSE existence is possible or not for Property V or even further.

[15]Since affiliation contains independence as a special case, the results can be *qualitatively* different, but must have an overlap.

[16]Both the revenue ranking under affiliation and the revenue equivalence theorem requires symmetry, risk neutrality and the same payoff by the lowest type of bidders.

[17]Recently, Monteiro and Moreira (2006) obtained further generalizations of equilibrium existence for non-affiliated variables. Their results are not directly related to positive dependence properties.

If we define $\Pi(x,y) = (x - b(y)) F(y|x)$, where $b(\cdot)$ is a candidate for the symmetric equilibrium,[18] then equilibrium existence is equivalent to $\Pi(x,x) \geq \Pi(x,y)$. Since $b(\cdot)$ is monotonic, one may conjecture that the monotonicity of $F(y|x)$ — as Property V assumes — may be sufficient to equilibrium existence, through some single crossing arguments (see Athey 2001). Since property V is still a strong property of positive dependence, this conjecture is reasonable. It turns out that the conjecture is wrong. The following theorem clarifies that MPSE existence does not generalize beyond Property VI.

Theorem 7 *If $f : [0,1]^2 \to \mathbb{R}$ satisfies property VI′, there is a symmetric pure strategy monotonic equilibrium. Nevertheless, property V is not sufficient for equilibrium existence.*
Proof. *See the appendix.* ■

Although there were reasons to expect MPSE existence for Property V, as discussed above, one can think that the non-existence is not too surprising, because it requires the inequality $\Pi(x,x) \geq \Pi(x,y)$ to be satisfied for every pair of points$(x,y) \in [0,1]^2$. Simply, there are many opportunities to break the equilibrium existence conditions. Nevertheless, the next implication — $R_2 \geq R_1$ — does not have this problem, because it is a comparison over expected values, that is, over integrals. Even if the inequality could be wrong for some realizations, it should be true in average for the cases of positive dependence. Thus, one could have the intuition that the revenue ranking implication should be stable across the cases of positive dependence.

There is yet another way of reaching the same conclusion: it is the intuition for the revenue ranking $R_2 \geq R_1$, which is a contribution of Klemperer (2004, p. 48-9):

> [In a first-price auction,] A player with value $v + dv$ who makes the same bid as a player with a value of v will pay the same price as a player with a value of v when she wins, but because of affiliation she will expect to win a bit less often [than in the case of independence]. That is, her higher signal makes her think her competitors are also likely to have higher signals, which is bad for her expected profits.
>
> But things are even worse in a second-price afilliated private-values auction for the buyer. Not only does her probability of winning diminish, as in the first-price auction, but her costs per victory are higher. This is because affiliation implies that contingent on her winning the auction, the higher her value the higher expected second-highest value which is the price she has to pay. Because the person with the highest value will win in either type of auction they are both equally efficient, and therefore the higher consumer surplus in first-price auction implies higher seller revenue in the second-price auction.

[18] This candidate is increasing and unique, as we show in section 4.

This intuition appeals mainly to the notion of positive dependence. Thus, the intuition should lead us to believe that the revenue ranking is still valid under other definitions of positive dependence. Despite these intuitive arguments, however, the following theorem shows that the implication $R_2 \geq R_1$ is not robust to weaker definitions of positive dependence. As before, Property V is again not sufficient for this result.

Theorem 8 *If f satisfies Property VI' (see definition above), then the second price auction gives greater revenue than the first price auction $(R_2 \geq R_1)$. Specifically, the revenue difference is given by*

$$\int_0^1 \int_0^x b^{'}(y) \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\, dy \cdot f(x)\, dx$$

where $b(\cdot)$ is the first price equilibrium bidding function, or by

$$\int_0^1 \int_0^x \left[\int_0^y L(\alpha|y)\, d\alpha \right] \cdot \left[1 - \frac{F(y|x)}{f(y|x)} \cdot \frac{f(y|y)}{F(y|y)} \right] \cdot f(y|x)\, dy \cdot f(x)\, dx, \qquad (3)$$

where $L(\alpha|t) = \exp\left[-\int_\alpha^t \frac{f(s|s)}{F(s|s)} ds \right]$. Moreover, Property V is not sufficient for this revenue rank.
Proof. *See the appendix.* ■

The results of this subsection are essentially negative: affiliation's implications are not robust. Nevertheless, this conclusion can change if we accept a weaker sense for the implications. This is to require the implications to be true in *average*, not for distributions obeying specific properties. In this *weaker sense*, the first implication — MPSE existence — is still not true, but $R_2 \geq R_1$ is. These results are reported in sections 4 and 5.

Now, we need to summarize and discuss our findings.

3.5 Discussion about the use affiliation in auction theory

In this section, we have shown that affiliation is restrictive, does not capture the positive dependence case (which implies that the given intuition may be misleading), and that some of its implications are not robust. Nevertheless, there are some economic models where it can be safely assumed and some of the implications can be true in more general cases, but in a weaker sense. How these two sides of affiliation's assessment compare?

First, for economic situations that can be well described by the models in subsection 3.3, affiliation is justified and its implications are ensured. As long one can accept the intuitions described there, there is absolutely no problem in doing auction theory with affiliation, despite the observation about restrictiveness. As long as we are in a setting in which affiliation is justified, we are free of all perils.

It is useful to illustrate the importance of this point by examining the situations where affiliation (under other names) is used in other sciences. For instance, affiliation is used in statistics, as Positive Likelihood Ratio Dependence (PLRD), the name given by Lehmann (1966) when he introduced the concept. In reliability theory, affiliation is generally referred to as Total Positivity of order two (TP_2) for the case of two variables or Multivariate Total Positivity of Order 2 (MTP_2) for n variables, after Karlin (1968). MTP_2 is used when the problem in study has some natural distributions and these distributions satisfy the MTP_2 condition. An example of this can be seen in the historical notes of Barlow and Proschan (1965) about reability theory. It is natural to assume that the failure rates of components or systems follow specific probabilistic distributions (exponentials, for instance) and such special distributions have the TP_2 property. Thus, the corresponding theory of total positive distributions can be advantegously used. Another example of this is the use of copulas. If we assume that the distribution is in a family of copulas that have the MTP property, then the use of affiliation's properties and implications is advantageous and are justified, by the choice of the set of distribution functions analyzed, as we discussed in subsection 3.3.[19]

However, we shall remember that the random variables (types) in auction theory represent information gathered by the bidders. There are some situations where we can assume special forms of the types' distributions (as the cases described in subsection 3.3), but in general there is no justification for assuming specific distributions. In fact, they are rarely assumed in the theory. Thus, there are meaningful and important economic situations that are not covered by affiliation.

From this, we conclude the following: (1) affiliation is useful as a theoretical tool, and can be safely assumed in some economic models; (2) affiliation does not cover all economically relevant cases and its implications may be not true in these cases; and, (3) there is need for considering more general approachs to dependence in auctions. The next section proposes an instance of such an approach.

4 A method that allows computational experiments

The complexity of auction models requires new tools for dealing with dependence and symmetry. For instance, even for single object auctions and independent types, if there is asymmetry between the bidders, it is not possible to obtain a complete characterization of the equilibrium strategies (see Lebrun 2006). Also under symmetry, but dependent types, there is not a developed theory beyond affiliation. If we want to treat asymmetry and general dependence, the conclusions seem to be beyond the reach of purely theoretical results.

In subsection 4.1 we argue that the auction phenomena related to dependence can be modeled and analyzed by considering a simpler but sufficiently rich class of distributions, which we introduce there.

Working in this class, we are able to completely characterize the PSE existence question in subsection 4.2. In this subsection, we also show that the proportion of distributions with PSE is small in the set of all densities considered.

[19]Li, Paarsch and Hubbard (2007) use copulas to model dependence in auction theory. They are able to find evidence of correlation between the bids.

4.1 The class of distributions

Modeling types as continuous real variables is a widespread practice in auction theory. The reason for that is clear: continuous variables allow the use of the convenient tools of calculus, such as derivatives and integrals, to obtain precise characterizations and uniqueness results. This is a very important advantaged, that should not be underestimated. (See Remark 12 below for a consequence of this). On the other hand, working with a continuous of types requires to rely only on analytical arguments to establish equilibrium results. As we show below, the set of first-price auctions with dependence where the standard arguments are able to establish pure strategy equilibrium existence is small in the set of all cases. Thus, continuous variables bring a benefit of characterization at the cost of loosing generality. We offer a method that has both advantages: it gives precise characterizations and is as general as an economist needs.[20] The idea is as follows.

Observe that the value of the single object in the auction is expressed up to cents and is obviously bounded. Thus, the number of actual possible values is finite. Nevertheless, instead of sticking to the (actual) case of discrete values, we allow them to be continuous, but impose, on the other hand, that the density functions are simple (see Figure 1 in the Introduction). In fact, it is sufficient to consider the particular set of simple symmetric functions $\mathcal{D}^k$, as defined in subsection 3.1. A density f in $\mathcal{D}^k$ can be described by a matrix, as the figure 3 below illustrate.

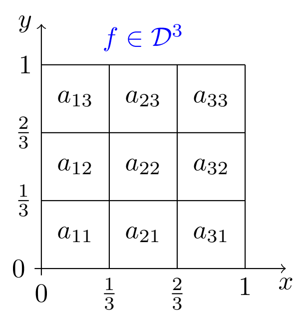


Figure 3: A density $f \in \mathcal{D}^k$ can be represented by a matrix $A = (a_{ij})$.

The restriction to $\mathcal{D}^\infty = \cup_{k \in \mathbb{N}} \mathcal{D}^k$ is a mathematical restriction that implies no economic restriction to the problem we are studying. Note also that the closure $\overline{\mathcal{D}^\infty}$ is the set of all densities $\mathcal{D}$. Now we describe how the equilibrium existence problem can be completely solved in the set $\mathcal{D}^\infty$.

First, recall the standard result of auction theory on PSE in private value auctions: if there is a differentiable symmetric increasing equilibrium, it satisfies the differential equation (see Krishna 2002 or Menezes and Monteiro 2005):

$$b'(t) = \frac{t - b(t)}{F(t|t)} f(t|t).$$

[20] The costs go to the complexity of the tools that are needed in the background.

If f is Lipschitz continuous, one can use Picard's theorem to show that this equation has a unique solution and, under some assumptions (basically, Property VI' of the previous subsection), it is possible to ensure that this solution is, in fact, equilibrium. Now, for $f \in \mathcal{D}^{\infty}$, the right hand side of the above equation is not continuous and one cannot directly apply Picard's theorem. We proceed as follows.

First, we show that if there is a symmetric increasing equilibrium b, under mild conditions (satisfied by $f \in \mathcal{D}^{\infty}$), b is continuous. We also prove that b is differentiable at the points where f is continuous. Thus, for $f \in \mathcal{D}^{\infty}$, b is continuous everywhere and differentiable everywhere but, possibly, at the points of the form $\frac{m}{k}$. See figure 4.

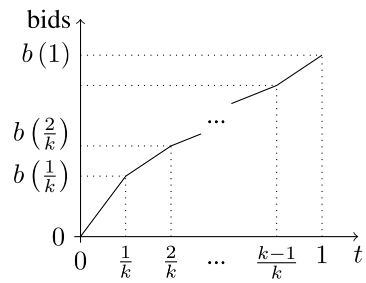


Figure 4: Bidding function for $f \in \mathcal{D}^{k}$.

With the initial condition $b(0) = 0$ and the above differential equation being valid for the first interval $\left(0, \frac{1}{k}\right)$, we have uniqueness of the solution on this interval and, thus, a unique value of $b\left(\frac{1}{k}\right)$. Since b is continuous, this value is the initial condition for the interval $\left(\frac{1}{k}, \frac{2}{k}\right)$, where we again obtain a unique solution and the uniqueness of the value $b\left(\frac{2}{k}\right)$. Proceeding in this way, we find that there is a unique b which can be a symmetric increasing equilibrium for an auction with $f \in \mathcal{D}^{\infty}$. In the supplement to this paper we prove the following:

Theorem 9 *Assume that u is twice continuously differentiable, $u' > 0$, $f \in \mathcal{D}^{k}$, f is symmetric and positive ($f > 0$). If $b : [0,1] \to \mathbb{R}$ is a symmetric increasing equilibrium, then b is continuous in $(0,1)$ and is differentiable almost everywhere in $(0,1)$ (it is may be non-differentiable only in the points $\frac{m}{k}$, for $m = 1, ..., k$). Moreover, b is the unique symmetric increasing equilibrium. If $u(x) = x^{1-c}$, for $c \in [0,1)$, b is given by*

$$b(x) = x - \int_{0}^{x} \exp\left[-\frac{1}{1-c}\int_{\alpha}^{x}\frac{f(s \mid s)}{F(s \mid s)}ds\right] d\alpha. \tag{4}$$

Proof. *See the supplement to this paper.* ■

Having established the uniqueness of the candidate for equilibrium, our task is reduced to verifying whether this candidate is, indeed, an equilibrium. We complete this task in the next subsection.

Even if the reader insists on considering the more general set of p.d.f.'s $\mathcal{D}$ — being aware that this is a matter of mathematical generality, but not of economic generality —

our set $\mathcal{D}^\infty$ is still dense in $\mathcal{D}$ and, thus, may arbitrarily approximate any conceivable p.d.f. in $\mathcal{D}$. In fact, the following result shows that equilibrium existence in the set $\mathcal{D}^\infty$ is sufficient for equilibrium existence in $\mathcal{D}$. This provides an additional justification of the method.

Proposition 10 *Let $f \in \mathcal{D}$ be continuous and symmetric. If $T^k(f)$ has a differentiable symmetric pure strategy equilibrium for all $k \geq k_0$, then so does f, and it is the limit of the equilibria of $T^k(f)$ as k goes to infinity.*[21]
Proof. *See the supplement to this paper.* ■

4.2 Equilibrium existence results

In the previous subsection, we established the uniqueness of the candidate for symmetric increasing equilibrium for $f \in \mathcal{D}^\infty$. Let $b(\cdot)$, given by (4) with $c = 0$, denote such a candidate. Let $\Pi(y, b(x)) = (y - b(x)) F(x|y)$ be the interim payoff of a player with type y who bids as type x, when the opponent follows $b(\cdot)$. Let $\Delta(x, y)$ represent $\Pi(y, b(x)) - \Pi(y, b(y))$. It is easy to see that $b(\cdot)$ is equilibrium if and only if $\Delta(x, y) \leq 0$ for all x and $y \in [0,1]^2$. Thus, the content of the next theorem is that it is possible to prove equilibrium existence by checking this condition only for a finite set of points:

Theorem 11 *Let $f \in \mathcal{D}^\infty$ be symmetric and strictly positive. There exists a finite set $P \subset [0,1]^2$ (precisely characterized in the supplement to this paper) such that f has a SMPSE if and only if $\Delta(x, y) \leq 0$ for all $(x, y) \in P$. Moreover, for $f \in D^k$, the verification of SMPSE can be done in $O(k^2)$ steps, that is, in polynomial time.*
Proof. *See the supplement to this paper.* ■

Remark 12 *It is important to compare this result with the faster known algorithms for solving simpler games as bimatrix games: they run in* exponential *time (see Savani and von Stengel 2006). This allows one to realize the important benefits of working with continuous variables but density functions in $\mathcal{D}^k$, as we propose. The characterization of the strategies obtained through differential equations allows one to drastically reduce the search for equilibrium candidates. Then, the fact that the density function is in $\mathcal{D}^k$ permit verify if the candidate is indeed an equilibrium in a very fast way. This allows auction theorists to run simulations for a big number of trials and get a good figure of what happens in the general.*

It is useful to say that the theorem is not trivial, since $\Delta(x, y)$ is not monotonic in the squares $\left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$. Indeed, the main part of the proof is the analysis of the non-monotonic function $\Delta(x, y)$ in the sets $\left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$ and the determination of its maxima for each of these sets. It turns out that we need to check a different number of points (between 1 and 5) for some of these squares.

Using Theorem 11, we can classify whether or not there is equilibrium, and, through numerical simulations, obtain the proportion of cases with pure strategy equilibrium.

[21] See the definition of T^k in subsection 3.1.

That is, for each trial $f \in \mathcal{D}^k$, we test whether the auction with bidders' types distributed according to f has a symmetric increasing pure strategy equilibrium. The results are shown in the Table 2 below.

For each k, 100%=distributions with equilibrium.

Distribution satisfying	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
Prop. VII (affiliation)	7.7%	0.07%	$< 10^{-7}$	–	–
Prop. VI	1.4%	0.08%	$\sim 10^{-6}$	$< 10^{-7}$	–
Prop. V	5.5%	0.75%	0.01%	$< 10^{-6}$	$< 10^{-7}$
Prop. IV	8.8%	4.1%	0.8%	0.1%	$\sim 10^{-5}$
$\sim$ Prop. IV	76.6%	95%	99.2%	99.9%	100%

Table 2 - Proportion of $f \in \mathcal{D}^k$ with PSE, satisfying properties IV-VII.

Table 2 shows that affiliation is restrictive even in the set of p.d.f.'s with symmetric increasing equilibrium. It is useful to record this fact separately. For this, let us introduce some useful notations. Let μ and μ^k denote the natural measures defined over $\mathcal{D}^\infty = \cup_{k=1}^{\infty} \mathcal{D}^k$ and $\mathcal{D}^k$, respectively, as constructed in the supplement to this paper and let $\mathcal{P}$ and $\mathcal{P}^k$ denote the set of p.d.f.'s in $\mathcal{D}^\infty$ and $\mathcal{D}^k$, respectively, which have a symmetric increasing pure strategy equilibrium. From the above table, we extract the following:

Observation 13 *Let $\mu^k \left(\cdot|\mathcal{P}^k\right)$ denote the measure induced by μ^kin the set $\mathcal{D}^k \cap \mathcal{P}^k$. Then, we have $\mu^k \left(\mathcal{A} \cap \mathcal{D}^k|\mathcal{P}^k\right) \downarrow 0$.*

Another way to say this is: there are many more cases with pure strategy equilibrium than affiliation allows us to prove.

Unfortunately, however, the set of p.d.f.'s with symmetric increasing equilibrium is also small. This result is proved formally in the following:

Theorem 14 *The measure of the set of densities $f \in \mathcal{D}^k$ which has PSE goes to zero as k increases, that is $\mu^k \left(\mathcal{P}^k\right) \downarrow 0$. Consequently, the measure of the set of densities $f \in \mathcal{D}^\infty$ with PSE is zero, that is, $\mu(\mathcal{P}) = 0$.*
Proof. *See the supplement to this paper.* ■

The proof of this theorem follows a simple idea: the equilibrium existence depends on a series of inequalities, the number of which increases with k. Although some care is needed for rigorously establishing the result, this simple observation is the heart of the argument. This gives us the intuition that the equilibrium constraints defining equilibrium increase faster than the degrees of freedom of the problem, when k increases.

The following table provides the numbers that come from numerical simulations and show that the convergence of $\mu^k \left(\mathcal{P}^k\right)$ to zero is also very fast.

	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
With PSE	43.3%	22.2%	11.4%	5.6%	2.7%	1.3%	0.6%
Without PSE	56.7%	77.8%	88.7%	94.4%	97.3%	98.7%	99.4%

Table 3 - Proportion of $f \in \mathcal{D}^k$ with and without SMPSE.

The result summarized in Table 3 is negative in the sense that it suggests that the focus on symmetric increasing equilibrium may be too narrow. Nevertheless, this is not yet sufficient to conclude that most of the equilibria are in mixed strategies. In fact, while we know that mixed strategy equilibria always exist (Jackson and Swinkels 2005), there is the possibility — not considered in our results — that there are equilibria in asymmetric or non-monotonic pure strategies.

Remark 15 *The fact that SMPSE existence is a restrictive property in the set of all distributions may lead to think that we should consider mixed strategy equilibria, instead.*

5 The Revenue Ranking of Auctions

Many results in auction theory (e.g. Maskin and Riley 1984, 2000), suggest that it is not possible to give simple predictions about the revenue ranking of auctions. The reason is that the expected difference of revenues can have any sign and the description of the set of distributions which has one sign or another is very difficult.

Fortunately, a simple characterization is available without further assumptions — if we work in a higher level of abstraction. To describe what we mean by this, consider the following figure.

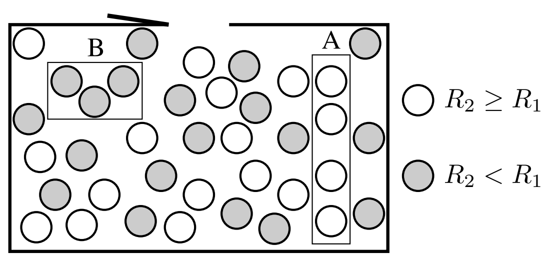


Figure 5: Illustrative representation of the set of distributions with any kind of dependence.

The standard approach to the problem of revenue ranking of auctions is to give conditions (the sets A and B above) under which the revenue ranking is defined. Such an approach is illustrated by Milgrom and Weber (1982)'s affiliation assumption and by Maskin and Riley (2000)'s three different assumptions for asymmetrical independent distributions.[22] Under each of the these assumptions, the authors are able to say precisely what the revenue ranking (the color of balls in the above figure) is. Now, if we think of the set of distributions as a black box or an urn, and we want a "prevision" of the color of the ball that we are going to extract from it (the expected revenue ranking), the "correct" type of answer to our problem is the characterization of the *probability* of obtaining one ranking or another.

[22]The reader should not be confused with the citation of Maskin and Riley (2000), which do not exactly deal with our problem — the revenue ranking with dependent types — but rather with the problem of revenue ranking with asymmetries. We cite them only to illustrate what we are calling the "standard approach" to the revenue ranking problem (in general).

Since we have the expression of the expected revenue difference, given by (3), we can obtain $\Delta_R^f = R_2^f - R_1^f$ and $r = \frac{R_2^f - R_1^f}{R_2^f}$, for each f. Generating a uniform sample of $f \in \mathcal{D}^k$, we can obtain the probabilistic distribution of Δ_R^f or of r. The procedure to generate $f \in \mathcal{D}^k$ uniformly is described in the supplement to this paper. The results are shown in subsection 5.1 below.

Moreover, we can also obtain theoretical results about what happens for $\mathcal{D}^N$ for a large N and even for $\mathcal{D}^\infty = \cup_{k=1}^\infty \mathcal{D}^k$. Nevertheless, for the last case, one has to be careful with the meaning of the "uniform" distribution. In the supplement to this paper we show that a natural measure can be defined for $\mathcal{D}^\infty$, which is analogous to Lebesgue measure, although it cannot have all the properties of the finite dimensional Lebesgue measure.

As such, this context-free approach, without specific information, allows one to obtain theoretical results and previsions based on simulations. One possible objection to this approach is that it considers too equally the p.d.f.'s in the sets $\mathcal{D}^k$. But this is exactly what we mean by "context-free". If one has information on the environment where the auction runs, so that one can restrict the set of suitable p.d.f.'s, then the uniform measure (of the context-free approach) should be substituted by the empirical measure obtained from the real-world applications.

Now, we present the results that one can obtain using this approach.

5.1 Results on Revenue Ranking

In the supplement to this paper, we develop the expression of the revenue differences from the second price auction to the first price auction for $f \in \mathcal{D}^k$. Let us denote by R_2^f the expected revenue (with respect to $f \in \mathcal{D}^k$) of the second price auction. Similarly, R_1^f denotes the expected revenue (with respect to $f \in \mathcal{D}^k$) of the first price auction. When there is no need to emphasize the p.d.f. $f \in \mathcal{D}^k$, we write R_1 and R_2 instead of R_1^f and R_2^f. Below, μ refers to the natural measure defined over $\mathcal{D}^\infty = \cup_{k=1}^\infty \mathcal{D}^k$, as further explained in the supplement to this paper. We observe the following fact in the simulations made:

Observation 16 *The expectation of the (expected) revenue differences, $R_2 - R_1$, is non-negative, that is, $E_\mu \left[R_2^f - R_1^f | f \in \mathcal{P}^k \right] \geq 0$, where $\mathcal{P}^k$ denotes the set of those $f \in \mathcal{D}^k$ for which there is a PSE in the first price auction.*[23]

The simulations were made as follows. We generated the distributions $f \in \mathcal{D}^k$ as described in the supplement to this paper. (It is the same process used in subsection **??** and section 4). We evaluate the revenue difference percentage, given by:

$$r = \frac{R_2^f - R_1^f}{R_2^f} \cdot 100\%,$$

that is, we carried out the following:

[23] This was verified for $k \leq 10$, but seems to be valid for larger k's.

Numerical experiments

In what follows, we will treat the numerical simulations as giving an "experimental distribution" of r. No confusion should arise between the "experimental distribution" of r and the distributions generated by each $f \in \mathcal{D}^k$. We generated 10^7 distributions $f \in \mathcal{D}^k$, for $k = 3, ..., 9$ and obtained r for each such f. The "experimental distribution" of r is characterized by the table below. It is worth saying that the results are already stable for 10^6 trials.

Distribution:	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
Expectation	4.5%	8.0%	10.3%	12.1%	13.4%	14.6%	15.5%
Variance	5.3%	6.9%	7.3%	7.2%	7.0%	6.8%	6.6%
5% quantile	-4%	-3%	-2%	0%	1%	3%	4%
10% quantile	-2%	-1%	0%	2%	3%	4%	6%
25% quantile	0%	2%	4%	6%	6%	8%	8%
50% quantile	2%	6%	8%	10%	10%	12.5%	12.5%
75% quantile	6%	10%	12.5%	15%	15%	17.5%	17.5%
90% quantile	10%	15%	17.5%	17.5%	19%	19%	19%
96% quantile	12.5%	17.5%	20%	20%	20%	20%	20%
99% quantile	15%	20%	25%	25%	25%	25%	25%

Table 4 - Expectation of the relative revenue differences (r) for $f \in \mathcal{D}^k$ with PSE.

Figure 6 shows the "experimental density" (histogram) of r for $k = 4$.

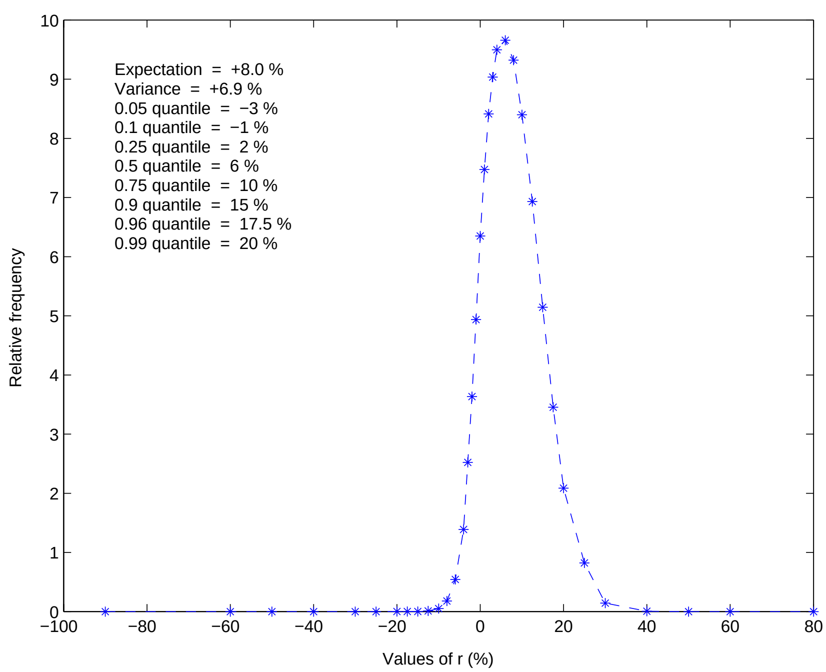


Figure 6: Histogram of r for $k = 4, c = 0$ — for those $f \in \mathcal{D}^k$ with PSE.

In Table 4, we displayed the results only for those f with PSE. If we consider all distributions, with and without PSE, we obtain the results in Table 5 below. This shows that the restriction of PSE existence matters for the distribution of r.

Distribution	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
Expectation	-0.08%	0.13%	0.28%	0.38%	0.46%	0.53%	0.57
Variance	10.5%	10.9%	10.5%	9.9%	9.4%	9.0%	8.5%
5% quantile	-25%	-20%	-20%	-17.5%	-17.5%	-15%	-15%
10% quantile	-15%	-15%	-15%	-15%	-12.5%	-12.5%	-12.5%
25% quantile	-8%	-8%	-8%	-8%	-8%	-8%	-8%
50% quantile	-1%	-1%	-1%	-1%	-1%	-1%	-1%
75% quantile	4%	6%	6%	6%	4%	4%	4%
90% quantile	10%	12.5%	12.5%	10%	10%	10%	10%
96% quantile	15%	15%	15%	15%	15%	15%	12.5%
99% quantile	25%	25%	25%	25%	25%	25%	25%

Table 5 - Expectation of the relative revenue differences (r) for all cases (with and without PSE).

The following table allows one to compare the effects of dependence and risk aversion to the expected revenue differences. For this, we restrict ourselves to the case of CRRA bidders, that is, bidders with utility function $u(x) = x^{1-c}$, where $c \in [0, 1)$.[24]

Expect.	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$c=0$	4.6%	8.0%	10.3%	12.1%	13.4%	14.6%	15.4%
$c=0.05$	3.7%	6.9%	8.8%	10.0%	10.8%	11.3%	11.6%
$c=0.1$	3.0%	5.7%	7.2%	8.0%	8.4%	8.6%	8.6%
$c=0.15$	2.2%	4.5%	5.7%	6.2%	6.3%	6.3%	6.3%
$c=0.2$	1.4%	3.4%	4.1%	4.4%	4.4%	4.4%	4.2%
$c=0.3$	0.0%	1.0%	1.2%	1.2%	1.1%	1.1%	1.0%
$c=0.4$	-1.4%	-1.5%	-1.8%	-1.8%	-1.8%	-1.7%	-1.5%
$c=0.52$	-2.9%	-4.7%	-5.3%	-5.2%	-4.9%	-4.6%	-4.1%
$c=0.65$	-3.7%	-8.8%	-9.6%	-9.3%	-8.6%	-7.9%	-7.1%
$c=0.8$	-6.3%	-14.7%	-15.9%	-15.2%	-14.1%	-13.1%	-12.1%

Table 6 - Expectation of the relative revenue differences (r) for bidders with CRRA function $u(x) = x^{1-c}$, where $c \in [0, 1)$.

6 Related literature, the contribution and future work

A few papers have pointed out restrictions or limitations to the implications of affiliation. Perry and Reny (1999) presented an example of a multi-unit auction where

[24]In Table 6, we restrict our study to the cases where PSE exists for $c = 0$. We do not have a generalization of the PSE existence result (Theorem 11) — and, thus, we do not have a procedure to test for PSE existence — for $c > 0$. The results in Table 6 should be considered with this in mind.

the linkage principle fails and the revenue ranking is reversed, even under affiliation. Thus, their criticism seems to be restricted to the generalization of the consequences of affiliation to multi-unit auctions. In contrast, we considered single-unit auctions and non-affiliated distributions.

Klemperer (2003) argues that, in real auctions, affiliation is not as important as asymmetry and collusion. He illustrates his arguments with examples of the 3G auctions conducted in Europe in 2000-2001.

Nevertheless, much more was written in accordance with the conclusions of affiliation. McMillan (1994, p.152) says that the auction theorists working as consultants to the FCC in spectrum auctions, advocated the adoption of an open auction using the linkage principle (Milgrom and Weber 1982a) as an argument: "Theory says, then, that the government can increase its revenue by publicizing any available information that affects the licensee's assessed value". The disadvantages of the open format in the presence of risk aversion and collusion were judged "to be outweighed by the bidders' ability to learn from other bids in the auction" (p. 152). Milgrom (1989, p. 13) emphasizes affiliation as the explanation of the predominance of the English auction over the first price auction.

This paper presents evidence that affiliation is a restrictive assumption. After developing an approach to test the existence of symmetric increasing pure strategy equilibrium (PSE) for simple density functions, we are able to verify that many cases with PSE do not satisfy affiliation. Also, the superiority of the English auction is not maintained even for distributions satisfying strong requirements of positive dependence. Nevertheless, we show that the original conclusion of Milgrom and Weber (1982a) (that positive dependence implies that English auctions gives higher revenue than first price auction) is true for a much larger set of cases , but in a weaker sense — "on average".

We would also like to highlight two conceptual contributions of this paper that may go beyond the actual applications made here.

One of these is the restriction to a simple space of distributions. The proposed space makes possible the complete characterization of the symmetric increasing pure strategy equilibrium existence problem, because it requires only elementary (albeit lengthy) calculations. The set of distributions considered is as general as necessary for economic applications and seems suitable for modeling the asymmetric bidders case as well.

The second conceptual contribution is a way of looking at the problem of revenue ranking. We propose two levels of the answer: an abstract, theoretical level and a simulation-driven applied level.

The first level allows general theoretical conclusions that may be useful as general, context-free guidance. Using the second level, applied economists can reach case-specific conclusions which may be more accurate and valuable.

These ideas are applicable for more general setups than those pursued here. Not only equilibrium existence but also the revenue ranking can be investigated in contexts of n asymmetric bidders, interdependent values, risk aversion and multi-units. Although these generalizations seem feasible, they are by no means trivial.[25]

[25]The generalization from $n = 2$ to general n can be pursued, at least for the symmetric case, using the expressions developed in the supplement of this paper. Only, instead of considering the expressions of $f(x|y)$ and $F(x|y)$ as coming from a bivariate distribution, we could write the expressions as coming from n variables but expressing them, as we did, by a matrix. Some expressions would change, but the main

Nevertheless, the main complement to this work seems to pertain to the fields of econometrics and experimentation. This is to develop a method to characterize the dependence (sets of distributions) typical in each specific situation. With such a method, applied works could characterize what happens in specific auctions. For instance, it is likely that the kind of dependence that appears in mineral rights auctions, or in electricity market auctions, is different from that observed in art auctions.

If such a method is developed and if the correspondent specific characterizations are done — these are big if's — we could find a way to explain why some kinds of markets almost always use a specific auction format, as observed by Maskin and Riley (2000): "rarely is any given kind of commodity sold through more than one sort of auction. Thus, for example, art is nearly always auctioned off according to the English rules, whereas job contracts are normally awarded through sealed bids" (p. 413). Maskin and Riley (2000) comment that the Revenue Equivalence Theorem is not able to explain such case-specific uniformity. The same argument also applies to affiliation, which would predict the use of English auctions for every situation.

An auction theory model capable of general conclusions and context-specific calibrations, using simulations and theoretical analysis, could explain this. If not, at least it would be more realistic and, thus, more useful.

Appendix

Proof of Theorem 6.

It is obvious that $(III) \Rightarrow (II) \Rightarrow (I)$. The implication $(IV) \Rightarrow (III)$ is Theorem 4.3. of Esary, Proschan and Walkup (1967). The implication $(V) \Rightarrow (IV)$ is proved by Tong (1980), chap. 5, p. 80. Thus, we have only to prove that $(VI) \Rightarrow (V)$, since the implication $(VII) \Rightarrow (VI)$ is Lemma 1 of Milgrom and Weber (1982a). Assume that $H(y|x) \equiv \frac{f(y|x)}{F(y|x)}$ is non-decreasing in x for all y. Then, $H(y|x) = \partial_y [\ln F(y|x)]$ and we have

$$1 - \ln[F(y|x)] = \int_y^\infty H(s|x)\, ds \geqslant \int_y^\infty H(s|x')\, ds = 1 - \ln[F(y|x')],$$

if $x \geqslant x'$. Then, $\ln[F(y|x)] \leqslant \ln[F(y|x')]$, which implies that $F(y|x)$ is non-increasing in x for all y, as required by property V.

The counterexamples for each passage are given by Tong (1980), chap. 5, except those involving property (VI): $(V) \nRightarrow (VI)$, $(VI) \nRightarrow (VII)$. For the first counter example, consider the following symmetric and continuous p.d.f. defined on $[0,1]^2$:

$$f(x,y) = \frac{d}{1 + 4(y-x)^2}$$

results would remain valid. Thus, the most difficult generalization seems to be to the asymmetric case.

where $d = [\arctan(2) - \ln(5)/4]^{-1}$ is the suitable constant for f to be a p.d.f. We have the marginal given by

$$f(y) = \frac{k}{2}[\arctan 2(1-y) + \arctan 2(y)]$$

so that we have, for $(x,y) \in [0,1]^2$:

$$f(x|y) = 2\left[1 + 4(y-x)^2\right]^{-1}[\arctan 2(1-y) + \arctan 2(y)]^{-1},$$

$$F(x|y) = \frac{[\arctan 2(x-y) + \arctan 2(y)]}{\arctan 2(1-y) + \arctan 2(y)}$$

and

$$\frac{F(x|y)}{f(x|y)} = 2\left[1 + 4(y-x)^2\right][\arctan(2x-2y) + \arctan(2y)].$$

Observe that for $y' = 0.91 > y = 0.9$ and $x = 0.1$,

$$\frac{F(x|y')}{f(x|y')} = 0.366863 > 0.366686 = \frac{F(x|y)}{f(x|y)},$$

which violates property (VI). On the other hand,

$$\begin{aligned}\partial_y[F(x|y)] &= \frac{\frac{2}{1+4y^2} - \frac{2}{1+4(x-y)^2}}{\arctan(2-2y) + \arctan(2y)} \\ &\quad - \frac{[\arctan(2x-2y) + \arctan(2y)]\left[\frac{2}{1+4y^2} - \frac{2}{1+4(1-y)^2}\right]}{[\arctan(2-2y) + \arctan(2y)]^2}\end{aligned}$$

In the considered range, the above expression is non-positive, so that property (V) is satisfied. Then, $(V) \not\Rightarrow (VI)$.

Now, fix an $\varepsilon < 1/2$ and consider the symmetric density function over $[0,1]^2$:

$$f(x,y) = \begin{cases} k_1, & \text{if } x + y \leqslant 2 - \varepsilon \\ k_2, & \text{otherwise} \end{cases}$$

where $k_1 > 1 > k_2 = 2\left[1 - k_1\left(1 - \varepsilon^2/2\right)\right]/\varepsilon^2 > 0$ and $\varepsilon \in (0, 1/2)$. For instance, we could choose $\varepsilon = 1/3$, $k_1 = 19/18$ and $k_2 = 1/18$. The conditional density function is given by

$$f(y|x) = \begin{cases} 1, & \text{if } x \leqslant 1 - \varepsilon \\ \frac{k_1}{k_2(x+\varepsilon-1)+k_1(2-\varepsilon-x)}, & \text{if } x > 1 - \varepsilon \text{ and if } y \leqslant 2 - \varepsilon - x \\ \frac{k_2}{k_2(x+\varepsilon-1)+k_1(2-\varepsilon-x)}, & \text{otherwise} \end{cases}$$

and the conditional c.d.f. is given by:

$$F\left(y|x\right)=\begin{cases}1, & \text{if } x\leqslant 1-\varepsilon\\ \frac{k_1 y}{k_2(x+\varepsilon-1)+k_1(2-\varepsilon-x)}, & \text{if } x>1-\varepsilon \text{ and if } y\leqslant 2-\varepsilon-x\\ \frac{k_2(y+x+\varepsilon-2)+k_1(2-\varepsilon-x)}{k_2(x+\varepsilon-1)+k_1(2-\varepsilon-x)}, & \text{otherwise}\end{cases}$$

and

$$\frac{F\left(y|x\right)}{f\left(y|x\right)}=\begin{cases}1, & \text{if } x\leqslant 1-\varepsilon\\ y, & \text{if } x>1-\varepsilon \text{ and if } y\leqslant 2-\varepsilon-x\\ y+x+\varepsilon-2+k_1/k_2\left(2-\varepsilon-x\right), & \text{otherwise}\end{cases}$$

Since $1-k_1/k_2<0$, the above expression is non-increasing in x for all y, so that property (VI) is satisfied. On the other hand, it is obvious that property (VII) does not hold:

$$f\left(0.5,0.5\right)f\left(1-\frac{\varepsilon}{2},1-\frac{\varepsilon}{2}\right)=k_2k_1<k_1^2=f\left(0.5,1-\frac{\varepsilon}{2}\right)f\left(0.5,1-\frac{\varepsilon}{2}\right).$$

This shows that $(VI)\nRightarrow(VII)$.■

Proof of Theorem 2

First, we prove that $C\backslash A$ is open. If $f\in C\backslash A$, then $f\left(x\right)f\left(x'\right)>f\left(x\wedge x'\right)f\left(x\vee x'\right)$, for some $x,x'\in[0,1]^n$. Fix such x and x' and define $K=f\left(x\right)+f\left(x'\right)+f\left(x\wedge x'\right)+f\left(x\vee x'\right)>0$. Choose $\varepsilon>0$ such that $2\varepsilon K<f\left(x\right)f\left(x'\right)-f\left(x\wedge x'\right)f\left(x\vee x'\right)$ and let $B_\varepsilon\left(f\right)$ be the open ball with radius ε and centre in f. Thus, if $g\in B_\varepsilon\left(f\right)$, $\|f-g\|<\varepsilon$, which implies $g\left(x\right)>f\left(x\right)-\varepsilon$, $g\left(x'\right)>f\left(x'\right)-\varepsilon$, $g\left(x\wedge x'\right)<f\left(x\wedge x'\right)+\varepsilon$, $g\left(x\vee x'\right)<f\left(x\vee x'\right)+\varepsilon$, so that

$$\begin{aligned}
& g\left(x\right)g\left(x'\right)-g\left(x\wedge x'\right)g\left(x\vee x'\right)\\
> & \left[f\left(x\right)-\varepsilon\right]\left[f\left(x'\right)-\varepsilon\right]-\left[f\left(x\wedge x'\right)+\varepsilon\right]\left[f\left(x\vee x'\right)+\varepsilon\right]\\
= & f\left(x\right)f\left(x'\right)-f\left(x\wedge x'\right)f\left(x\vee x'\right)-\varepsilon\left[f\left(x\right)+f\left(x'\right)+f\left(x\wedge x'\right)+f\left(x\vee x'\right)\right]\\
= & f\left(x\right)f\left(x'\right)-f\left(x\wedge x'\right)f\left(x\vee x'\right)-\varepsilon K\\
> & \varepsilon K>0,
\end{aligned}$$

which implies that $B_\varepsilon\left(f\right)\subset C\backslash A$, as we wanted to show.

Now, let us show that $C\backslash A$ is dense, that is, given $f\in C$ and $\varepsilon>0$, there exists $g\in B_\varepsilon\left(f\right)\cap C\backslash A$. Since $f\in C$, it is uniformly continuous (because $[0,1]^n$ is compact), that is, given $\eta>0$, there exists $\delta>0$ such that $\|x-x'\|_{\mathbb{R}^n}<2\delta$ implies $|f\left(x\right)-f\left(x'\right)|<\eta$. Take $\eta=\varepsilon/4$ and the corresponding δ.

Choose $a\in(4\delta,1-4\delta)$ and define x (x') by specifying that their first $\left\lfloor\frac{n}{2}\right\rfloor$ coordinates are equal to $a-\delta$ $(a+\delta)$ and the last ones to be equal to $a+\delta$ $(a-\delta)$. Thus, $x\wedge x'=(a-\delta,...,a-\delta)$ and $x\vee x'$ = $(a+\delta,...,a+\delta)$. Let x_0 denote the vector $(a,...,a)$. For $y = x$, x', $x \wedge x'$ or $x \vee x'$, we have: $|f(y) - f(x_0)| < \eta$. Let $\xi : (-1,1)^n \to \mathbb{R}$ be a smooth function that vanishes outside $\left(-\frac{\delta}{2}, \frac{\delta}{2}\right)^n$ and equals 1 in $\left(-\frac{\delta}{4}, \frac{\delta}{4}\right)^n$. Define the function g by

$$\begin{aligned} g(y) &= f(y) + 2\eta\xi(y - x) + 2\eta\xi(y - x') \\ &\quad -2\eta\xi(y - x \wedge x') - 2\eta\xi(y - x \vee x'). \end{aligned}$$

Observe that $\|g - f\| = 2\eta = \varepsilon/2$, that is, $g \in B_\varepsilon(f)$. In fact, $g \in B_\varepsilon(f) \cap C \backslash A$, because

$$\begin{aligned} g(x) &= f(x) + 2\eta > f(x_0) + \eta; \\ g(x') &= f(x) + 2\eta > f(x_0) + \eta; \\ g(x \wedge x') &= f(x \wedge x') - 2\eta < f(x_0) - \eta; \\ g(x \vee x') &= f(x \vee x') - 2\eta < f(x_0) - \eta, \end{aligned}$$

which implies

$$\begin{aligned} &g(x)\, g(x') - g(x \wedge x')\, g(x \vee x') \\ > \; &[f(x_0) + \eta]^2 - [f(x_0) - \eta]^2 \\ = \; &4\eta > 0. \blacksquare \end{aligned}$$

Proof of Theorem 7

The equilibrium existence follows from Milgrom and Weber (1982a)'s proof. For the counterexample, consider the p.d.f. defined in the proof of Theorem 6:

$$f(x,y) = \frac{d}{1 + 4(y-x)^2},$$

where $d = [\arctan(2) - \ln(5)/4]^{-1}$. In the proof of Theorem 6, we established that this p.d.f. satisfies Property V but not Property VI and that:

$$F(x|y) = \frac{[\arctan 2(x-y) + \arctan 2(y)]}{\arctan 2(1-y) + \arctan 2(y)}.$$

From Theorem 9, it is sufficient to prove that

$$b(y) = y - \int_0^y \exp\left[-\frac{1}{2}\int_z^y \frac{1}{\arctan 2w} dw\right] dz$$

cannot be an equilibrium, that is, to verify the existence of x and y such that

$$(y - b(y))\, F(y|y) < (y - b(x))\, F(x|y).$$

This simplifies to the condition:

$$\frac{\int_0^y \exp\left[-\frac{1}{2}\int_z^y \frac{1}{\arctan 2w}dw\right]dz}{y-x+\int_0^x \exp\left[-\frac{1}{2}\int_z^x \frac{1}{\arctan 2w}dw\right]dz} < \frac{\arctan 2\left(x-y\right)}{\arctan 2y}+1.$$

Let $y = 0.5$ and $x = 1$. Mathematica gives $\int_0^y \exp\left[-\frac{1}{2}\int_z^y \frac{1}{\arctan 2w}dw\right]dz = 0.391128$ and $\int_0^x \exp\left[-\frac{1}{2}\int_z^x \frac{1}{\arctan 2w}dw\right]dz = 0.745072$. Thus, we have:

$$\frac{0.391128}{-0.5+0.745072} = 1.59597 < 2 = \frac{\arctan 2\left(x-y\right)}{\arctan 2y}+1,$$

which concludes the verification for the counterexample of PSE existence.

Proof of Theorem 8

The dominant strategy for each bidder in the second price auction is to bid his value: $b^2\left(t\right)=t$. Then, the expected payment by a bidder in the second price auction, P^2, is given by:

$$\begin{aligned} P^2 &= \int_{\left[\underline{t},\bar{t}\right]}\int_{\left[\underline{t},x\right]} yf\left(y|x\right)dy\cdot f\left(x\right)dx = \\ &= \int_{\left[\underline{t},\bar{t}\right]}\int_{\left[\underline{t},x\right]} \left[y-b\left(y\right)\right]f\left(y|x\right)dy\cdot f\left(x\right)dx + \int_{\left[\underline{t},\bar{t}\right]}\int_{\left[\underline{t},x\right]} b\left(y\right)f\left(y|x\right)dy\cdot f\left(x\right)dx, \end{aligned}$$

where $b\left(\cdot\right)$ gives the equilibrium strategy for symmetric first price auctions. Thus, the first integral can be substituted by $\int_{\left[\underline{t},\bar{t}\right]}\int_{\left[\underline{t},x\right]} b^{'}\left(y\right)\frac{F\left(y|y\right)}{f\left(y|y\right)}f\left(y|x\right)dy\cdot f\left(x\right)dx$, from the first order condition: $b^{'}\left(y\right)=\left[y-b\left(y\right)\right]\frac{f\left(y|y\right)}{F\left(y|y\right)}$. The last integral can be integrated by parts, to:

$$\begin{aligned} &\int_{\left[\underline{t},\bar{t}\right]}\int_{\left[\underline{t},x\right]} b\left(y\right)f\left(y|x\right)dy\cdot f\left(x\right)dx \\ =& \int_{\left[\underline{t},\bar{t}\right]}\left[b\left(x\right)F\left(x|x\right)-\int_{\left[\underline{t},x\right]} b^{'}\left(y\right)F\left(y|x\right)dy\right]\cdot f\left(x\right)dx \\ =& \int_{\left[\underline{t},\bar{t}\right]} b\left(x\right)F\left(x|x\right)\cdot f\left(x\right)dx - \int_{\left[\underline{t},\bar{t}\right]}\int_{\left[\underline{t},x\right]} b^{'}\left(y\right)F\left(y|x\right)dy\cdot f\left(x\right)dx \end{aligned}$$

In the last line, the first integral is just the expected payment for the first price auction, P^1. Thus, we have

$$
\begin{aligned}
D &= P^2 - P^1 \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b'(y) \frac{F(y|y)}{f(y|y)} f(y|x)\, dy \cdot f(x)\, dx \\
&\quad - \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b'(y) F(y|x)\, dy \cdot f(x)\, dx \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b'(y) \left[\frac{F(y|y)}{f(y|y)} f(y|x) - F(y|x) \right] dy \cdot f(x)\, dx \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b'(y) \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\, dy \cdot f(x)\, dx
\end{aligned}
$$

Remember that $b(t) = \int_{[\underline{t},t]} \alpha dL(\alpha|t) = t - \int_{[\underline{t},t]} L(\alpha|t)\, d\alpha$, where $L(\alpha|t) = \exp\left[-\int_\alpha^t \frac{f(s|s)}{F(s|s)} ds \right]$. So, we have

$$
\begin{aligned}
b'(y) &= 1 - L(y|y) - \int_{[\underline{t},y]} \partial_y L(\alpha|y)\, d\alpha \\
&= \frac{f(y|y)}{F(y|y)} \int_{[\underline{t},y]} L(\alpha|y)\, d\alpha.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
D &= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} \frac{f(y|y)}{F(y|y)} \int_{[\underline{t},y]} L(\alpha|y)\, d\alpha \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\, dy \cdot f(x)\, dx \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} \left[\int_{[\underline{t},y]} L(\alpha|y)\, d\alpha \right] \cdot \left[1 - \frac{F(y|x)}{f(y|x)} \cdot \frac{f(y|y)}{F(y|y)} \right] \cdot f(y|x)\, dy \cdot f(x)\, dx
\end{aligned}
$$

This is the desired expression. For the counterexample, consider the matrix

$$
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1.6938 & 0.3812 & 0.4140 \\ 0.3812 & 2.1318 & 0.5817 \\ 0.4140 & 0.5817 & 2.4206 \end{bmatrix},
$$

and define the p.d.f. as follows:

$$
f(x,y) = a_{mp} \text{ if } (x,y) \in \left(\frac{m-1}{k}, \frac{m}{k} \right] \times \left(\frac{p-1}{k}, \frac{p}{k} \right],
$$

for $m, p \in \{1,2,3\}$ and $k = 3$. This distribution satisfies property V (but not property VI) and has a pure strategy equilibrium. The expected revenue from a second price auction is 0.4295, while the expected revenue of a first price auction is 0.4608, which is nearly 7% above.

■

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