Vertical differentiation with non-uniform consumers' distribution

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Abstract

The note explores a vertical differentiation model with a continuous non-uniform consumers' distribution. First a result concerning the finiteness property obtained with uniform consumers' distribution is generalized. Second we prove an existence result of price equilibrium when the distribution is concave. Finally we exhibit a counter-example to the existence of price equilibrium to show that the concavity assumption is not superfluous.

Keywords: Vertical differentiation, non-uniform distribution, finiteness property, price equilibrium existence.

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1 Introduction

The vertical differentiation model introduced by Mussa and Rosen (1978) has been studied with two types of consumers' distribution: discrete, for instance by Acharyya (1998) and Kim and Kim (1996); and uniform continuous distribution, for instance by Anderson, de Palma and Thisse (1992, p.305-316) and Lahmandi-Ayed (2000, 2004). We aim in this paper to relax these two polar hypotheses by considering the case of continuous non-uniform consumers' distribution. The main object of the paper is to explore the existence of price equilibria in pure strategies.

We first deal with the finiteness property, which means that the number of active firms at price equilibrium is upper-bounded independently of qualities choice, even in the absence of fixed cost, in which case the market is said to be a natural oligopoly. We generalize the result concerning the finiteness property obtained in the case of costless production with uniform consumers' distribution, to the case of non-uniform distributions.

However this result is proved at price equilibrium. Hence to make sense, it must have cases of application, i.e. the equilibrium existence issue must be explored. We prove that the concavity of the density function is sufficient to imply the existence of a price equilibrium for any chosen qualities since the profits are then quasi-concave. Finally, to show that the concavity of the distribution is not superfluous, we exhibit a counter-example to the existence of price equilibrium with a non-concave density function.

The note is organized as follows. Section 2 describes the model. Section 3 considers the finiteness property. Section 4 provides the existence result and the counter-example.

2 The model

We consider the model first proposed by Mussa and Rosen (1978) (used afterwards by Anderson, de Palma and Thisse and numerous authors) and try to follow closely the same notation. The utility of a consumer depends on the price and the quality of the product. Each consumer is supposed to buy one unit of the product from the firm that ensures to her the highest utility, except if all the prices exceed her income. Consumers are supposed to have the same income y but are different in their intensity of preference for quality.

Suppose that there are n firms each selling a distinct variant. Note p_i and q_i respectively the price and the quality of variant i ($i = 1, ..., n$) where $q_i \in [\underline{q}, \overline{q}]$ for all $i = 1, ..., n$. $\underline{q} > 0$ is the minimal quality accepted. For a n-uple of qualities $(q_1, ..., q_n)$, qualities are always ordered such that $q_1 < q_2 < ... < q_n$. When firm i charges price p_i, we denote by p_{-i} the prices of the other firms.

The conditional indirect utility of a consumer of type θ buying one unit of variant i is given by:

$$V_i(\theta) = y - p_i + \theta q_i, \tag{1}$$

Consumers are supposed to be distributed with a density function $h(\theta)$ on the segment $[\underline{\theta}, \overline{\theta}]$, with h twice continuously differentiable. This is the main difference with the previous papers on the issue, which have only considered a constant distribution or a discrete one.

We suppose $\underline{\theta} > 0$. This implies that all consumers agree on the ranking of the variants. If all qualities are sold at the same price, consumers will choose to buy the highest quality. A firm is said to be *active* if it faces a positive demand. Rationality imposes that active firms have non-negative profits at equilibrium price since a firm has always the possibility of exiting. This implies that an active firm has a positive price. Hence if n firms are active with $q_1 < q_2 < .... < q_n$, we have necessarily $p_1 < p_2 < ..... < p_n$.

Since prices are more easily adjustable than qualities, it is reasonable to model price and quality competition by a two-step game in which qualities are chosen simultaneously in the first step and prices are chosen simultaneously in the second one, after having observed the choice of the first step. A price equilibrium is thus the Nash equilibrium of the subgame corresponding to the choice of prices at given qualities. We can naturally suppose that the strategy spaces in prices are upper-bounded by y.

To focus on the introduction of non-uniform density, we suppose that production is costless.

3 Finiteness property

In this section we generalize some results previously obtained on the finiteness property in the case of uniform consumers' distribution, to the case of non-uniform one. We only assume that the distribution is positive.

Proposition 1 *If the density function satisfies $h(\theta) > 0, \ \forall\theta \in [\underline{\theta}, \overline{\theta}]$ then the market is a natural oligopoly, i.e. the number of active firms at price equilibrium is limited by an upper-bound independent of quality choice.*

This is a generalization of the result obtained in the case of costless production with uniform consumers' distribution (Anderson et al.). Note that the uniform distribution is obviously positive. The proof uses exactly the same technique as in the case of uniform distribution and is given in Appendix. In the same way, we can generalize the result obtained by Anderson et al. in the case of production cost with constant returns, if the density function is positive. The result is not included in the paper as the existence results are provided only for costless production.

4 Equilibrium existence

The results on the finiteness property easily generalize to the case of non uniform density, as we observed in the preceding section. But the demonstrated results make sense only if they have cases of application. We devote this section to this object exploring the existence issue. We first prove that the concavity of the density function implies the existence of a price equilibrium. Then, to show that the concavity assumption is not superfluous, we exhibit a counter-example.

4.1 An existence result

Proposition 2 *Suppose that h is concave and $h(\theta) > 0$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$, then there exists an equilibrium in prices whatever the chosen qualities.*

The following lemma is needed to prove Proposition 2.

Lemma 1 *Let f be a continuously differentiable function on some closed interval $[a, b]$. Suppose that for all $x \in [a, b]$, $f(x) = 0$ implies $f'(x) < 0$ (respectively $f'(x) > 0$). Then f is equal to zero at most once on $[a, b]$.*

Note that by derivability at a we mean the right derivative and at b the left one[1].

Proof of Proposition 2. The profit functions are obviously continuous. We are going to prove that they are quasi-concave so that the equilibrium existence is a consequence of standard existence results of Nash equilibria. Below (including first, second and third steps), we give the intuitive arguments underlying the proof, then (in the final step), we provide a more precise mathematical formulation.

A firm is either always inactive, in which case, the profit is obviously quasi-concave, or it is active until an upper-bound and then inactive. In the second case, the profit is positive and then equal to 0. To prove the quasi-concavity of the profit function, it is enough to prove it on the interval where the firm is active.

An active firm may be in one of the three following situations: It has two active neighbors, it has only one active neighbor (the right or the left side one), it is alone on the market. To prove the quasi-concavity of the profit function, we will apply a technique already used in Shaked and Sutton (1983)[2]. First, we are going to prove that the profit function is quasi-concave in each situation on the whole interval where it is active. Second, we show that at a critical price where the situation changes, the left side derivative is greater than the right side derivative.

We begin with the simplest case. When a firm is alone on the market (for low values of prices), its profit given by $p_i(\overline{\theta} - \underline{\theta})$ is an increasing function, hence a quasi-concave function.

First step. The profit of firm i on the interval of prices where its closest neighbors are active, is quasi-concave.

We denote by $\hat{\theta}_j$ the marginal consumer between firm $j-1$ and firm j and by $\delta_j = q_j - q_{j-1}$. We have:

$$\hat{\theta}_j = \frac{p_j - p_{j-1}}{\delta_j}.$$

Let us consider a firm $i = 2, .., n-1$ such that $i-1$ and $i+1$ are respectively the closest left side and the closest right side active competitors. In this first step, we consider the profit of firm i, π_i, on the segment of prices p_i such that $\hat{\theta}_{i-1} < \hat{\theta}_i < \hat{\theta}_{i+1}$,

[1] Lemma 1 is a consequence of a more general result on degree theory (see for instance Milnor, 1965) or it can be proved by elementary argument of real analysis.

[2] A real function defined on an interval I, which admits left and right side derivatives everywhere, is quasi-concave if it is so on each sub-interval and if the left side derivative is greater than the right side derivative at each limit point between two sub-intervals.

(i.e. such that both neighbors are active) and prove that π_i is quasi-concave on this segment.

Let $\tilde{p} = \frac{\delta_i p_{i+1} + \delta_{i+1} p_{i-1}}{\delta_i + \delta_{i+1}}$ be the upper-bound of the interval on which the profit function is studied. We have $\pi_i(\tilde{p}) = 0$.

Considering the profit as a function of price p_i,

$$\pi_i = p_i \int_{\hat{\theta}_i}^{\hat{\theta}_{i+1}} h(\theta) d\theta$$

The three first derivatives of π_i w.r.t. p_i are given by:

$$\pi_i'(p_i) = \int_{\hat{\theta}_i}^{\hat{\theta}_{i+1}} h(\theta) d\theta + p[-\frac{h(\hat{\theta}_{i+1})}{\delta_{i+1}} - \frac{h(\hat{\theta}_i)}{\delta_i}]$$

$$\pi_i''(p_i) = -\frac{2h(\hat{\theta}_{i+1})}{\delta_{i+1}} - \frac{2h(\hat{\theta}_i)}{\delta_i} + p_i(\frac{h'(\hat{\theta}_{i+1})}{\delta_{i+1}^2} - \frac{h'(\hat{\theta}_i)}{\delta_i^2})$$

$$\pi_i'''(p_i) = 3[\frac{h'(\hat{\theta}_{i+1})}{\delta_{i+1}^2} - \frac{h'(\hat{\theta}_i)}{\delta_i^2}] + p[-\frac{h''(\hat{\theta}_{i+1})}{\delta_{i+1}^3} - \frac{h''(\hat{\theta}_i)}{\delta_i^3}]$$

Since h is always positive, $\pi_i''(p) = 0$ implies $\frac{h'(\hat{\theta}_{i+1})}{\delta_{i+1}^2} - \frac{h'(\hat{\theta}_i)}{\delta_i^2} > 0$, which in turn implies $\pi_i'''(p) > 0$ since $h'' \leq 0$ (as h is concave). Applying Lemma 1, π_i'' is equal to zero at most once on the considered interval. If ever this occurs at $\hat{p}$, we have $\pi_i'''(\hat{p}) > 0$. Hence π_i is always concave, always convex or concave then convex.

Moreover $\pi_i(\tilde{p}) = 0$ and $\pi_i(p_i) > 0$ on the considered interval, which implies $\pi'(\tilde{p}) \leq 0$. Hence π_i is decreasing on the interval where it is convex (whenever it exists) thus is necessarily quasi-concave.

Second step. The profit is quasi-concave on the interval of prices where the firm has only one neighbor.

We only give the computations for an active right side competitor, since the other case may be dealt with in an analogous way. Consider the profit

$$\pi_i = p_i \int_{\underline{\theta}}^{\hat{\theta}_{i+1}} h(\theta) d\theta$$

$$\pi_i'(p_i) = \int_{\underline{\theta}}^{\hat{\theta}_{i+1}} h(\theta) d\theta - \frac{p_i}{\delta_{i+1}} h(\hat{\theta}_{i+1})$$

$$\pi_i''(p_i) = -\frac{2h(\hat{\theta}_{i+1})}{\delta_{i+1}} + \frac{p_i h'(\hat{\theta}_{i+1})}{\delta_{i+1}^2}$$

$$\pi_i'''(p_i) = \frac{3h'(\hat{\theta}_{i+1})}{\delta_{i+1}^2} - \frac{p_i h''(\hat{\theta}_{i+1})}{\delta_{i+1}^3}$$

We note that $\pi_i'' = 0$ implies $\pi_i''' > 0$ and we continue as in the first step using Lemma 1.

Third step. We now prove that when the demand expression changes, the left side derivative is greater than the right side one so that the profit is quasi-concave on the union of the two intervals.

To this respect, we have to distinguish three cases: When lowering its price,

$2 \to 2$ The firm having two active neighbors, has again two new active neighbors.

$2 \to 1$ The firm having two active neighbors, has now only one active neighbor.

$1 \to 0$ The firm having only one active neighbor, becomes alone on the market.

($2 \to 2$) Consider a firm $i = 3, .., n-2$. The demand changes when either $\hat{\theta}_i = \hat{\theta}_{i-1}$ in which case firm $i-1$ is eliminated, or $\hat{\theta}_{i+1} = \hat{\theta}_{i+2}$, in which case firm $i+1$ is eliminated.

Suppose that when lowering its price, firm i eliminates first firm $i-1$. The right side derivative is given by:

$$\frac{\partial}{\partial p_i}[p_i \int_{\hat{\theta}_i}^{\hat{\theta}_{i+1}} h(\theta)d\theta]_{\hat{\theta}_i=\hat{\theta}_{i-1}} = \int_{\hat{\theta}_i}^{\hat{\theta}_{i+1}} h(\theta)d\theta + p_i(-\frac{h(\hat{\theta}_{i+1})}{q_{i+1}-q_i} - \frac{h(\hat{\theta}_{i-1})}{q_i - q_{i-1}})$$

As for the left side derivative, firm $i-1$ disappears and firm i has firm $i-2$ as its left side neighbor. Denote by $\tilde{\theta}_{i,i-2}$ the marginal consumer between firm i and firm $i-2$.

$$\tilde{\theta}_{i,i-2} = \frac{p_i - p_{i-2}}{q_i - q_{i-2}}$$

The profit

$$\pi_i = p_i \int_{\tilde{\theta}_{i,i-2}}^{\hat{\theta}_{i+1}} h(\theta)d\theta$$

The derivative of this profit is given by:

$$\int_{\tilde{\theta}_{i,i-2}}^{\hat{\theta}_{i+1}} h(\theta)d\theta + p_i(-\frac{h(\hat{\theta}_{i+1})}{q_{i+1}-q_i} - \frac{h(\tilde{\theta}_{i,i-2})}{q_i - q_{i-2}})$$

But when $\hat{\theta}_i = \hat{\theta}_{i-1}$, we have also $\tilde{\theta}_{i,i-2} = \hat{\theta}_{i-1}$. Hence the left side derivative at this price equals:

$$\int_{\hat{\theta}_{i-1}}^{\hat{\theta}_{i+1}} h(\theta)d\theta + p_i(-\frac{h(\hat{\theta}_{i+1})}{q_{i+1}-q_i} - \frac{h(\hat{\theta}_{i-1})}{q_i - q_{i-2}})$$

The simple fact that $q_i - q_{i-2} > q_i - q_{i-1}$ implies that the left side derivative is greater than the right side derivative. The same type of computation holds when firm i eliminates first its right side neighbor $i+1$.

($2 \to 1$) Suppose now that firm i lowers its price until it eliminates its left side neighbor and has no more a left side neighbor.

The right side derivative remains the same as previously, i.e.

$$\int_{\underline{\theta}}^{\hat{\theta}_{i+1}} h(\theta)d\theta + p_i(-\frac{h(\hat{\theta}_{i+1})}{q_{i+1}-q_i} - \frac{h(\underline{\theta})}{q_i - q_{i-1}})$$

The left side derivative now is:

$$\int_{\underline{\theta}}^{\hat{\theta}_{i+1}} h(\theta)d\theta - \frac{p_i h(\hat{\theta}_{i+1})}{q_{i+1}-q_i}$$

Hence the left side derivative is greater than the right side derivative. Calculations are the same if firm i has no more a right side neighbor.

$(1 \rightarrow 0)$ It remains now to prove that the expression change keeps the quasi-concavity of the function when the firm while lowering its price eliminates the only remaining neighbor. Suppose the remaining neighbor is the right side one so that the right side expression of the profit is given by:

$$\pi_i = p_i \int_{\underline{\theta}}^{\hat{\theta}_{i+1}} h(\theta)d\theta,$$

and the right side derivative exactly when the expression changes (i.e. $\hat{\theta}_{i+1} = \overline{\theta}$) is:

$$\int_{\underline{\theta}}^{\overline{\theta}} h(\theta)d\theta - \frac{p_i h(\overline{\theta})}{\delta_{i+1}}.$$

Whereas the right side expression is given by:

$$\pi_i = p_i \int_{\underline{\theta}}^{\overline{\theta}} h(\theta)d\theta,$$

so that the right side derivative given by

$$\int_{\underline{\theta}}^{\overline{\theta}} h(\theta)d\theta,$$

is larger than the right side one.

Final step. The previous arguments explaining how the expression change occurs follow closely the intuition and help understand intuitively how the proof works. In fact they stem from a general expression of the profit, which should allow a more elegant proof. Indeed, the profit of firm i may be written as follows:

$$\pi_i = p_i \int_{\underline{\theta}_i(p_i)}^{\overline{\theta}(p_i)} h(\theta)d\theta$$

with

$$\underline{\theta}_i(p_i) = \max\{\underline{\theta}, \frac{p_i - p_j}{q_i - q_j}, j = 1, .., i-1\}$$

and

$$\overline{\theta}_i(p_i) = \min\{\overline{\theta}, \frac{p_i - p_j}{q_i - q_j}, j = i+1, .., n\}$$

The expression of the profit changes each time either the expression of $\underline{\theta}_i(p_i)$ or of $\overline{\theta}_i(p_i)$ changes.

Note on the one hand that the function $\underline{\theta}_i(p_i)$ is increasing, continuous, piecewise linear and convex. The convexity in this case means that as the expression changes, the left side derivative is lower than the right side one. On the other hand, the function $\overline{\theta}_i(p_i)$ is decreasing, continuous, piecewise linear and concave. These properties secure the quasi-concavity of the profit. ■

4.2 A counter-example

We aim in this paragraph to show that the concavity of the density function is not superfluous in the existence theorem. In the counter-example we consider a duopoly with a non concave density function and show that no equilibrium exists.

We consider the duopoly case. Firm i sells quality q_i, $i = 1, 2$ with $q_1 < q_2$ at price p_i. Denote by $\delta = q_2 - q_1$ and by $\hat{\theta} = \frac{p_2 - p_1}{\delta}$ the marginal consumer between both firms. Consumers are distributed over $[\underline{\theta}, \overline{\theta}]$ with the density function:

$$h(\theta) = \frac{1}{\theta}.$$

We are going to prove that the parameters of the model may be adjusted so that no price equilibrium exists. Before stating precisely the result, we try to explain roughly why this may happen.

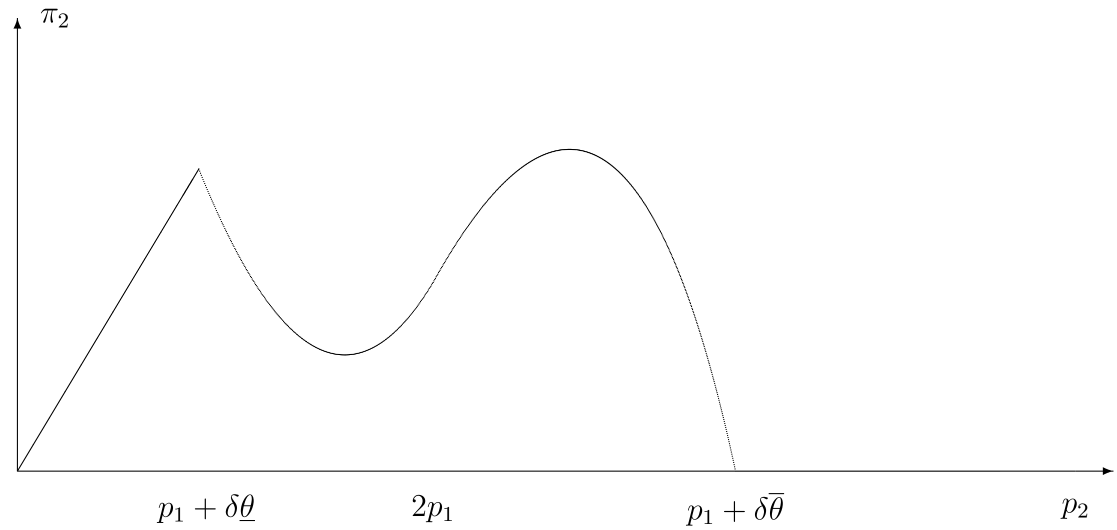
Consider firm 2. For prices p_2 such that $\underline{\theta} < \hat{\theta} < \overline{\theta}$, viz., $p_1 + \delta\underline{\theta} < p_2 < p_1 + \delta\overline{\theta}$, firm 2's profit is given by:

$$\pi_2 = p_2 \int_{\hat{\theta}}^{\overline{\theta}} h(\theta) d\theta$$

The second derivative of this profit w.r.t. p_2 writes:

$$\frac{\partial^2 \pi_2}{\partial p_2^2} = \frac{-p_2 + 2p_1}{(p_2 - p_1)^2}$$

We have $\frac{\partial^2 \pi_2}{\partial p_2^2} > 0$ only if $p_2 < 2p_1$. If ever $\tilde{p}_2 = 2p_1 \in [p_1 + \delta\underline{\theta}, p_1 + \delta\overline{\theta}]$, which is equivalent to $\delta\underline{\theta} < p_1 < \delta\overline{\theta}$, then π_2 is convex until $\tilde{p}_2$ then becomes concave. This case may be represented by the following figure.



In general π_2 is not quasi-concave and the best reply correspondence may not be convex-valued. This is why a price equilibrium may fail to exist. We now formally enounce and prove the result.

Proposition 3 *Suppose that the density function is given by: $h(\theta) = \frac{1}{\theta}$. Denote by $\alpha = \log(\frac{\overline{\theta}}{\underline{\theta}}) - 1$ and take $\alpha = 5$. Suppose finally that the parameters δ, $\underline{\theta}$ and y satisfy $\frac{(\alpha+2)}{2} e^{\alpha/2} \delta\underline{\theta} < y$.*

Then there is no equilibrium in prices.

Proof. We are going to prove that no price equilibrium exists in this case by examining the two possible types of equilibria:

- "non-interior" price equilibria involving an inactive firm.
- "interior" price equilibria, i.e. such that both firms are active.

1) "non-interior" equilibrium.

If firm i is inactive at equilibrium (p_1^*, p_2^*), this implies that its profit π_i is necessarily constant equal to zero for all prices when the price of its opponent is equal to p_{-i}^*, otherwise a profitable deviation exists. This cannot be the case of firm 2 because its profit is equal to $\pi_2 = p_2 N$ on the segment $[0, p_1 + \delta\underline{\theta}]$, which has a non empty interior for all prices $p_1 \geq 0$. If ever a firm is inactive at price equilibrium, it is necessarily firm 1 and firm 2 has all the market. Thus $\frac{p_2^* - p_1^*}{\delta} \leq \underline{\theta}$, so that all consumers prefer firm 2. But for prices $p_2 \leq p_1^* + \delta\underline{\theta}$, the profit function of firm 2 is strictly increasing w.r.t. p_2. Thus equilibrium prices satisfy necessarily

$$p_2^* = p_1^* + \delta\underline{\theta}$$

Suppose $p_1^* > 0$, then $p_2^* > \delta\underline{\theta}$. Then by deviating to some price $0 < p_1 = p_1^* - \epsilon < p_2^* - \delta\underline{\theta}$, firm 1 secures a positive demand and a positive profit. Hence necessarily $p_1^* = 0$ and $p_2^* = \delta\underline{\theta}$.

We now prove that this couple of prices cannot be an equilibrium. Suppose firm 1's price is fixed $p_1^* = 0$ and consider the possible deviations of firm 2. For all prices $p_2 \geq \delta\underline{\theta}$, the profit of firm 2 is given by:

$$\pi_2 = p_2 \int_{\frac{p_2}{\delta}}^{\overline{\theta}} \frac{d\theta}{\theta} = p_2 \log(\frac{\delta\overline{\theta}}{p_2})$$

The partial derivative of this profit w.r.t. p_2 writes:

$$\frac{\partial \pi_2}{\partial p_2} = \log(\frac{\delta\overline{\theta}}{p_2}) - 1$$

At $p_2 = \delta\underline{\theta}$, $\frac{\partial \pi_2}{\partial p_2} = \log(\frac{\overline{\theta}}{\underline{\theta}}) - 1 = \alpha > 0$
Hence firm 2 has profitable deviations.

2) "Interior" equilibria.

Suppose an "interior" price equilibrium exists. Necessarily $p_1 < p_2 \leq y$ and $\underline{\theta} < \hat{\theta} < \overline{\theta}$.

First step: We prove that the couples of prices (p_1, p_2) such that $p_1 < p_2 = y$ cannot be an equilibrium.

If such a price is an equilibrium, we must have

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \log(\frac{\hat{\theta}}{\underline{\theta}}) - \frac{p_1}{p_2 - p_1} = 0 \\ \\ \frac{\partial \pi_2}{\partial p_2} = \log(\frac{\overline{\theta}}{\hat{\theta}}) - \frac{p_2}{p_2 - p_1} \geq 0 \end{cases}$$

Replacing p_2 by y and rearranging the second equation yield:

$$\log(\frac{y - p_1}{\delta\underline{\theta}}) - \frac{p_1}{y - p_1} = 0 \text{ and } p_1 \leq \frac{\alpha}{2+\alpha} y.$$

Denote by

$$f(p_1) = \log(\frac{y - p_1}{\delta\underline{\theta}}) - \frac{p_1}{y - p_1} \text{ and by } \ \bar{p}_1 = \frac{\alpha}{2+\alpha} y.$$

We have $f'(p_1) < 0$, hence f is decreasing. Furthermore,

$$f(\bar{p}_1) = \log(\frac{2y}{(\alpha+2)\delta\underline{\theta}}) - \frac{\alpha}{2}.$$

The condition $\frac{(\alpha+2)}{2} e^{\alpha/2} \delta\underline{\theta} < y$ implies $f(\bar{p}_1) > 0$. Hence there is no price p_1 satisfying $p_1 \leq \bar{p}_1$ such that $f(p_1) = 0$. Thus under our hypotheses a couple of prices $p_1 < p_2 = y$ cannot be an equilibirum.

Second step: We now deal with prices $p_1 < p_2 < y$.

If such a couple of prices is an equilibrium, first order conditions hold and are equalities. Thus

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \log(\frac{\hat{\theta}}{\underline{\theta}}) - \frac{p_1}{p_2 - p_1} = 0 \\ \\ \frac{\partial \pi_2}{\partial p_2} = \log(\frac{\overline{\theta}}{\hat{\theta}}) - \frac{p_2}{p_2 - p_1} = 0 \end{cases}$$

Adding both equalities and arranging the equation yields

$$p_2 = p_1 \frac{(2+\alpha)}{\alpha}.$$

Replacing in $\frac{\partial \pi_1}{\partial p_1} = 0$ and then using the relation between both prices allow the calculation of (p_1^*, p_2^*) , the equilibrium candidate

$$p_1^* = \frac{\alpha}{2} e^{\alpha/2} \delta \underline{\theta}$$

and

$$p_2^* = \frac{(\alpha+2)}{2} e^{\alpha/2} \delta \underline{\theta}$$

Note that $p_2^* < y$.

We are now going to prove that such a couple of prices is not an equilibrium, by proving that firm 2 has a profitable deviation. Indeed we prove that the profit of firm 2 when it charges price $p_1^* + \delta\underline{\theta}$ is higher than its profit with price p_2^*.

With (p_1^*, p_2^*), we have $\hat{\theta} = e^{\alpha/2}\underline{\theta}$ and firm 2's profit

$$\pi_2^* = p_2^* \log(\frac{\overline{\theta}}{\hat{\theta}}) = e^{\alpha/2} \frac{(\alpha+2)^2}{4} \delta \underline{\theta};$$

while with the couple of prices $(p_1^*, p_2 = p_1^* + \delta\underline{\theta})$, we have $\hat{\theta} = \underline{\theta}$ and firm 2's profit

$$\bar{\pi}_2 = p_2 \log(\frac{\overline{\theta}}{\underline{\theta}}) = (\frac{\alpha}{2} e^{\alpha/2} + 1)(\alpha+1)\delta\underline{\theta}$$

Replacing $\alpha = 5$, we have $\bar{\pi}_2 > \pi_2^*$.

Hence there does not exist an interior equilibrium.

■

Appendix

Proof of Proposition 1. Suppose there are n active firms $i = 1, .., n$. Denote by

$$\hat{\theta}_i = \frac{p_i - p_{i-1}}{q_i - q_{i-1}}$$

the marginal consumer between firm i and firm $i-1$ for $i = 2, .., n$. For all firms to be active one must have:

$$\underline{\theta} < \hat{\theta}_2 < ... < \hat{\theta}_n < \overline{\theta}.$$

At price equilibrium, all prices are positive. Hence for an interior solution or price y to be the solution, first order conditions write:

$$\frac{\partial \pi_i}{\partial p_i} \geq 0$$

For firm $i = 2, .., n-1$, the profit is given by

$$\pi_i = p_i \int_{\hat{\theta}_i}^{\hat{\theta}_{i+1}} h(\theta) d\theta$$

First order conditions yield:

$$\begin{aligned}
\int_{\hat{\theta}_i}^{\hat{\theta}_{i+1}} h(\theta) d\theta \quad &\geq \quad \frac{p_i h(\hat{\theta}_{i+1})}{q_{i+1} - q_i} + \frac{p_i h(\hat{\theta}_i)}{q_i - q_{i-1}} &(2) \\
&> \quad \frac{p_i - p_{i-1}}{q_i - q_{i-1}} h(\hat{\theta}_i) &(3) \\
&\geq \quad \underline{\theta h} &(4)
\end{aligned}$$

where $\underline{h} = \min\{h(\theta) \mid \theta \in [\underline{\theta}, \overline{\theta}]\}$. But the market share of firm i, M_i, is given by

$$M_i = \int_{\hat{\theta}_i}^{\hat{\theta}_{i+1}} h(\theta) d\theta,$$

and

$$\sum_{i=2}^{n-1} M_i \leq \int_{\underline{\theta}}^{\overline{\theta}} h(\theta) d\theta.$$

Thus

$$(n-2)\underline{h\theta} \leq \int_{\underline{\theta}}^{\overline{\theta}} h(\theta) d\theta.$$

Since h is continuous, we have $\underline{h} > 0$.

The number of active firms n is thus necessarily upper-bounded by a limit independent of qualities choice. ∎

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