The Efficient Outcome in Multilateral Bargaining

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Introduction

- Bilateral bargaining: uniqueness under complete information (*Rubinstein*)
- Multilateral Bargaining: 1. Any efficient Outcome Can be Supported by SPE.
- 2. No aggreement is also SPE outcome.

Objectives of This Paper

- Construct a new approach to multilateral bargaining
- Generalize Rubinstein-Stahl model of bilateral bargaining to multilateral bargaining with different discount factors
- Produce a unique outcome for multilateral bargaining.

Our approach

1. Define bargaining operators for each player and for the game
2. Generate a Markov chain based on bargaining operators, regarding shares for each player as a measure over states
3. The unique invariant measure is the unique outcome of the bargaining game, which is the eigenvector associated with the 1 of the bargaining operator of the game

Outline

- Rubinstein-Stahl bilateral bargaining
- Multilateral bargaining
- Multilateral bargaining with randomized order of moves
  - model description
  - equilibrium characterization
  - examples (bilateral, trilateral with same discount factor)
- Conclusion

Rubinstein-Stahl bilateral bargaining

(backward induction)

- Two players make alternative offers to share a pie of size 1
- $T\,(= 0, ...2k-1)$ periods, δ is discount factor

t	player 1's share, x_1	player 2's share, x_2
$2k-1$	0	1
$2k-2$	$1-\delta_2$	δ_2
$2k-3$	$\delta_1(1-\delta_2)$	$1-\delta_1(1-\delta_2)$
$2k-4$	$1-\delta_2[1-\delta_1(1-\delta_2)]$	$\delta_2[1-\delta_1(1-\delta_2)]$
...	...	...
0	$\delta_1[1-\delta_2[1-\delta_1[1-...]]]$	$1-\delta_1[1-\delta_2[1-\delta_1[1-...]]]$

Rubinstein-Stahl bilateral bargaining

The unique subgame perfect equilibrium (Lemma 1 and Proposition 1)

$$x_1(T) = \frac{(1-\delta_2)\left(1-\delta_1^k\delta_2^k\right)}{1-\delta_1\delta_2}$$

$$x_2(T) = \frac{\delta_2(1-\delta_1)\left(1-\delta_1^k\delta_2^k\right)}{1-\delta_1\delta_2} + \delta_1^k\delta_2^k$$

as $T \to \infty$

$$x_1^* = \frac{1-\delta_2}{1-\delta_1\delta_2}$$

$$x_2^* = \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$$

Rubinstein-Stahl bilateral bargaining
(our approach)

- Define individual bargaining operators

$$B_1 = \begin{bmatrix} 1 & 1-\delta_2 \\ 0 & \delta_2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \delta_1 & 0 \\ 1-\delta_1 & 1 \end{bmatrix}$$

- Define bargaining operator for the game

$$B = B_1 B_2 = \begin{bmatrix} 1-\delta_2(1-\delta_1) & 1-\delta_2 \\ \delta_2(1-\delta_1) & \delta_2 \end{bmatrix}$$

- Proposition 2: *i) Let $x = (x_1, x_2) \in \Delta^2$ be an acceptable offer made by player j in period t, then $x' = B_i x$ will be an acceptable offer made by player i in period $t-1$, $i \neq j$. ii) If $x = (x_1, x_2) \in \Delta^2$ is an acceptable offer made by player i in period t, then ${x'}^{2k} = B^k x = (B_i B_j)^k x$ will be an acceptable offer made by the same player in period $t - 2k$.*

The bargaining outcome as a fixed point of the bargaining o
(our approach)

- Proposition 3

  i) The stationary subgame perfect equilibrium outcome $x^* = (\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2})$ *is the unique fixed point of* B *in* Δ^2.

  ii) Given any $x^0 \in \Delta^2$, $x^k = B^k x^0$ *converges to the unique SPE outcome* x^*.

Multilateral bargaining

(bargaining operators)

For player i:

$$B_i = \begin{bmatrix} \delta_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \delta_2 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1-\delta_1 & 1-\delta_2 & \dots & 1-\delta_{i-1} & 1 & 1-\delta_{i+1} & \dots & 1-\delta_n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & \delta_n \end{bmatrix}$$

For the game:

$$B = B_1 B_2 \dots B_n$$

Multilateral bargaining

(finite horizon equilibrium)

Proposition 5

i) If x *is an acceptable offer made by player* i *in period* t, *then* $x' = B_{i-1}x$ *is an acceptable offer made by player* $(i-1)$ *in period* $(t-1)$.

ii) $G(T)$ *has a unique subgame perfect Nash equilibrium outcome*

$$x^* = B^{[\frac{T}{n}]}B_1B_2...B_{l-1}\mathbf{1}_l,$$

where $T = l(\mathrm{mod}(n))$, $\mathbf{1}_l = (0, ..., 1, ..., 0)$, *and* $[\frac{T}{n}]$ *is the integer part of* $\frac{T}{n}$.

Multilateral bargaining
(infinite horizon equilibrium)

Proposition 6

i) There is a unique efficient subgame perfect equilibrium outcome $x^* \in \mathbf{\Delta}^n$ *for* $G(\infty)$.

ii) The unique efficient subgame perfect equilibrium outcome, x^*, *is a fixed point for* $B = B_1 \cdots B_n$ *in* $\mathbf{\Delta}^n$.

iii) For any $x^0 \in \mathbf{\Delta}^n$, *the sequence* $x^k = B^k x^0$ *converges to the unique efficient bargaining outcome* x^*.

Multilateral bargaining

(examples)

- Trilateral bargaining

1. Bargaining operators for individuals

$$B_1 = \begin{bmatrix} 1 & 1-\delta_2 & 1-\delta_3 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \delta_1 & 0 & 0 \\ 1-\delta_1 & 1 & 1-\delta_3 \\ 0 & 0 & \delta_3 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 1-\delta_1 & 1-\delta_2 & 1 \end{bmatrix}.$$

and for the game: $B = B_1 B_2 B_3$.

2. Calculate the eigenvector associated with unitary eigenvalue of B. It is B's unique fixed point in Δ^3, also the unique subgame perfect equilibrium outcome for the game.

- Trilateral bargaining equilibrium

with different discount factors

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} \frac{(1-\delta_2)(1-\delta_3)(1+\delta_3+\delta_2\delta_3)}{(1-\delta_1\delta_2\delta_3)^2+\delta_1\delta_3(\delta_2-\delta_3)+\delta_1\delta_2(\delta_3-\delta_1)+\delta_2\delta_3(\delta_1-\delta_2)} \\ \frac{\delta_2(1-\delta_1)(1-\delta_3)(1+\delta_1+\delta_1\delta_3)}{(1-\delta_1\delta_2\delta_3)^2+\delta_1\delta_3(\delta_2-\delta_3)+\delta_1\delta_2(\delta_3-\delta_1)+\delta_2\delta_3(\delta_1-\delta_2)} \\ \frac{\delta_3^2(1-\delta_1)(1-\delta_2)(1+\delta_2+\delta_1\delta_2)}{(1-\delta_1\delta_2\delta_3)^2+\delta_1\delta_3(\delta_2-\delta_3)+\delta_1\delta_2(\delta_3-\delta_1)+\delta_2\delta_3(\delta_1-\delta_2)} \end{bmatrix}$$

with same discount factor

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} \frac{1}{1+\delta+\delta^2} \\ \frac{\delta}{1+\delta+\delta^2} \\ \frac{\delta^2}{1+\delta+\delta^2} \end{bmatrix}$$

Multilateral bargaining
(examples)

- N-person bargaining with same discount factor

1. Bargaining operators for individuals

$$B_i = \begin{bmatrix} \delta & 0 & ... & 0 & 0 & 0 & ... & 0 \\ 0 & \delta & ... & 0 & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 1-\delta & 1-\delta & ... & 1-\delta & 1 & 1-\delta & ... & 1-\delta \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & 0 & 0 & ... & \delta \end{bmatrix}$$

and for the game:

$$B = B_1 B_2 ... B_n$$

$$B = \begin{bmatrix} 1-\delta+\delta^n & 1-\delta & 1-\delta & \dots & 1-\delta \\ \delta-\delta^2 & \delta-\delta^2+\delta^n & \delta-\delta^2 & \dots & \delta-\delta^2 \\ \delta^2-\delta^3 & \delta^2-\delta^3 & \delta^2-\delta^3+\delta^n & \dots & \delta^2-\delta^3 \\ \dots & \dots & \dots & \dots & \dots \\ \delta^{n-1}-\delta^n & \delta^{n-1}-\delta^n & \delta^{n-1}-\delta^n & \dots & \delta^{n-1}-\delta^n+\delta^n \end{bmatrix}$$

2. B's unique fixed point in Δ^n is the only subgame perfect equilibrium

$$x^* = \left(\frac{(1-\delta)}{1-\delta^n}, \frac{\delta(1-\delta)}{1-\delta^n}, \frac{\delta^2(1-\delta)}{1-\delta^n}, \dots, \frac{\delta^{n-1}(1-\delta)}{1-\delta^n} \right).$$

Multilateral bargaining

(close-form characterization of equilibrium)

- Partition player k's bargaining operator $B_k = \begin{bmatrix} B_{11}^{(k)} & B_{12}^{(k)} \\ B_{21}^{(k)} & B_{22}^{(k)} \end{bmatrix}$ and bargaining operator $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$, where $B_{11}^{(k)}$ and B_{11} are scalar matrix, $B_{22}^{(k)}$ and B_{22} are $(n-1) \times (n-1)$.
- $B_{22} = B_{22}^{(1)} B_{22}^{(2)} ... B_{22}^{(n)}$.
- Define share function $s(\delta_2, \delta_3, ..., \delta_n) = \det(I - B_{22})$, a polynomial function of $(\delta_2, \delta_3, ..., \delta_n)$.
- Let $s_k\left(\delta_1, .., \widehat{\delta}_k, ..., \delta_n\right) = s(\delta_{k+1}, ..., \delta_n, \delta_1, ..., \delta_{k-1})$ be the polynomial function obtained by rotating the variables.

- Lemma 2

If B is a stochastic matrix, then $\lim_{n\to\infty} B^n$ exists if and only if B is primitive. In this case, $\lim_{n\to\infty} B^n = \frac{A(1)}{c^{(1)}(1)}$, where $A(\lambda) = adj(\lambda I - B)$, the adjoint matrix for $(\lambda I - B)$, and $c(\lambda) = \det(\lambda I - B)$ is the characteristic polynomial for B, and $c^{(1)}(1)$ is the first order derivative evaluated at $\lambda = 1$.

- Proposition 7

In the unique efficient bargaining outcome $x^* = \left(x_1^*, ..., x_n^*\right)$, the share for player k can be represented as

$$x_k^* = \frac{\delta_k^{k-1} s_k\left(\delta_1, ..\widehat{\delta}_k, ..., \delta_n\right)}{c^{(1)}(1)},$$

where $s_k(\delta_1, ..., \widehat{\delta}_k, ..., \delta_n) = \det\left(I - B_{22}^{(k)} ... B_{22}^{(n)} B_{22}^{(1)} ... B_{22}^{(k-1)}\right)$.

Multilateral bargaining with randomized order of moves
(model description)

- Transition matrix $P = \left[p_{ij}\right]_{n \times n}$,
  p_{ij} :probability of shifting the right of offer from player i to player j.

- Offer matrix $X = [x_{\cdot 1}, ..., x_{\cdot n}]$, where $x_{\cdot i}$ is offer made by player i

$$x_{\cdot i} = \begin{bmatrix} x_{1i} \\ x_{2i} \\ \cdots \\ x_{ni} \end{bmatrix} \in \Delta^n$$

Multilateral bargaining with randomized order of moves

(equilibrium characterization)

- Suppose player i makes an offer in period t. Expected payoff in period $t+1$ for all players is

$$E_{P_i}X = XP_i^T,$$

where P_i is ith row of P. Backward induction implies that player i's acceptable offer in period t is

$$x_{\cdot i} = B_i X P_i^T, i = 1, 2, ..., n$$

The solution to above system of equations is the equilibrium.

- Let $vec(A)_{n^2\times 1}$ be the vector obtained by listing the columns of A one above another. Then $vec\left(B_i X P_i^T\right) = (P_i \otimes B_i)\, vec(X)$

- Proposition 8

  The unique outcome in the N-person bargaining problem with random transition P is equivalent to finding a column stochastic matrix X that satisfies the following system of equations:

$$\begin{aligned} x_{\cdot 1} &= P_1 \otimes B_1 vec(X), \\ x_{\cdot 2} &= P_2 \otimes B_2 vec(X), \\ &\cdots\cdot \\ x_{\cdot n} &= P_n \otimes B_n vec(X). \end{aligned} \qquad \text{Let } P \circledast B = \begin{bmatrix} P_1 \otimes B_1 \\ P_2 \otimes B_2 \\ \cdots \\ P_n \otimes B_n \end{bmatrix}_{n^2\times n^2}$$

- The equilibrium is a matrix $X \in \Delta^n \times \cdots \times \Delta^n$ such that

$$vec(X) = [P \circledast B]\, vec(X)$$

Multilateral bargaining with randomized order of moves

(examples)

- Bilateral bargaining

  Equilibrium outcome matrix X is characterized by

$$\begin{aligned}
\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} &= B_1 X P_1^T = \begin{bmatrix} 1 & 1-\delta_2 \\ 0 & \delta_2 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix}, \\
\begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} &= B_2 X P_2^T = \begin{bmatrix} \delta_1 & 0 \\ 1-\delta_1 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} p_{21} \\ p_{22} \end{bmatrix}.
\end{aligned}$$

- Bilateral bargaining equilibrium

with random transition

$$X = \begin{bmatrix} \frac{(1-\delta_2)(1-p_{22}\delta_1)}{1-\delta_1\delta_2+p_{11}\delta_2(\delta_1-1)+p_{22}\delta_1(\delta_2-1)} & \frac{\delta_1(1-\delta_2)(1-p_{22})}{1-\delta_1\delta_2+p_{11}\delta_2(\delta_1-1)+p_{22}\delta_1(\delta_2-1)} \\ \frac{\delta_2(1-\delta_1)(1-p_{11})}{1-\delta_1\delta_2+p_{11}\delta_2(\delta_1-1)+p_{22}\delta_1(\delta_2-1)} & \frac{(1-\delta_1)(1-p_{11}\delta_2)}{1-\delta_1\delta_2+p_{11}\delta_2(\delta_1-1)+p_{22}\delta_1(\delta_2-1)} \end{bmatrix}.$$

with alternative offers, $p_{11} = p_{22} = 0$, reduce to classical Rubinstein outcome

$$X = \begin{bmatrix} \frac{(1-\delta_2)}{1-\delta_1\delta_2} & \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2} \\ \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2} & \frac{(1-\delta_1)}{1-\delta_1\delta_2} \end{bmatrix}.$$

Summary

- A new approach to multilateral bargaining.

1. Define bargaining operators for each player and for the game
2. Generate a Markov chain based on bargaining operators, regarding shares for each player as a measure over states
3. The unique invariant measure is the unique equilibrium of the bargaining game, which is the eigenvector associated with the unitary eigenvalue of the bargaining operator of the game

- Generalize Rubinstein-Stahl model of bilateral bargaining to multilateral bargaining with different discount factors, circumventing intractability problem in backward induction.
- Generalize to multilateral bargaining with random transition of moves.