Each question is given equal weight in the grading. \( \mathbb{Q} \) denotes the rational numbers, \( \mathbb{R} \) the reals, and \( \mathbb{C} \) the complex numbers. Fields should not be assumed to be the reals or complex numbers unless specifically stated. You should justify all assertions to receive full credit. Demonstrating an in-depth understanding of the material is the most important factor.

1. Prove (without using the structure theorem for finite abelian groups) that the direct product of a cyclic group of order \( r \) with a cyclic group of order \( s \) is isomorphic to the direct product of a cyclic group of order \( m = \text{lcm}(r, s) \) with a cyclic group of order \( n = \gcd(r, s) \).

2. Let \( R = \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\} \) be the Gaussian integers \( (i^2 = -1) \). Let \( I \) be the principal ideal in \( R \) generated by \( i - 2 \). Prove that \( R/I \cong \mathbb{Z}/5\mathbb{Z} \).

3. Let \( V \) be a finite dimensional inner product space, and let \( f \in V^* \) be a linear functional. Prove that there is an element \( w \in V \) such that for all \( v \in V \), \( f(v) = (v|w) \).

4. Let \( V = \mathbb{C}^3 \), and let \( T : V \to V \) be the linear operator with matrix

\[
A = \begin{pmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 1 & -1
\end{pmatrix}
\]

with the standard basis of \( V \). Prove that \( T \) has no cyclic vector. Give the Jordan and rational normal forms for \( T \).

5. Let \( A \) be a 6 by 6 matrix over the complex numbers. Suppose that the minimal polynomial \( p(x) \) of \( A \) generates the ideal in \( \mathbb{C}[x] \) generated by the polynomials \( f = (x^2 + 1)(x - 1)^3x^2 \) and \( g = (x - 1)^2x^3(x + 1)^2 \). What are the possible Jordan normal forms for \( A \)?

6. Let \( F \) be a field, and let \( E \) be a field containing \( F \). Suppose that \( F \subset L_1 \subset E \) and \( F \subset L_2 \subset E \) are subfields of \( E \) containing \( F \). Let \( L \) be the smallest subfield of \( E \) containing both \( L_1 \) and \( L_2 \). Prove that

\[
[L : F] \leq [L_1 : F][L_2 : F].
\]

When does equality hold?
Qualifying Exam: August 2008

I. What we are looking for is whether you have understood some important concepts and whether you are able to apply the general theory in particular situations. We also want to test your computational skills.

II. Give as many details as possible. If you are invoking a theorem to prove something you must state that theorem precisely.

III. Q, R and C stand for the field of rational numbers, real numbers and complex numbers respectively.

(1) Show that a finite cyclic group of order \( n \) has exactly one subgroup of each order \( d \) dividing \( n \), and that these are all the subgroups it has. (15 points)

(2) (a) Find the number of irreducible quadratic polynomials in \( \mathbb{Z}_p[x] \), where \( p \) is a prime. (10 points)
(b) Show that for \( p \) a prime, the polynomial \( x^p + a \) in \( \mathbb{Z}_p[x] \) is not irreducible for any \( a \in \mathbb{Z}_p \). (10 points)

(3) (a) Find all fields that are finite extensions of \( \mathbb{R} \). (10 points)
(You may assume that \( \mathbb{C} \) is algebraically closed.)
(b) Prove that the algebraic closure of \( \mathbb{Q} \) in \( \mathbb{C} \) is not a finite extension of \( \mathbb{Q} \). (10 points)

(4) Let \( A = \begin{bmatrix} 3 & 2 & -3 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \) be a 3 \( \times \) 3 matrix.

(a) Find the characteristic polynomial and eigenvalues of \( A \). (5 points)
(b) Find the Jordan canonical form \( J \). (5 points)
(c) Find a matrix \( P \) so that \( J = P^{-1}AP \). (10 points)

(5) Let \( T : \mathbb{C}^n \to \mathbb{C}^n \) be a linear transformation. We say \( \mathbf{v} \in \mathbb{C}^n \) is a generalized eigenvector of \( T \) with corresponding eigenvalue \( \lambda \) if \( \mathbf{v} \neq \mathbf{0} \) and \( (T - \lambda I)^k(\mathbf{v}) = \mathbf{0} \) for some positive integer \( k \). Define the generalized \( \lambda \)-eigenspace

\[ E(\lambda) = \{ \mathbf{v} \in \mathbb{C}^n | (T - \lambda I)^k(\mathbf{v}) = \mathbf{0} \text{ for some positive integer } k \} \]

(a) Prove that \( E(\lambda) \) is a subspace of \( \mathbb{C}^n \). (5 points)
(b) Suppose \( \lambda_1, ..., \lambda_k \) are distinct scalars and \( \mathbf{v}_1, ..., \mathbf{v}_k \) are generalized eigenvectors of \( T \) with corresponding eigenvalues \( \lambda_1, ..., \lambda_k \) respectively. Prove that \( \{\mathbf{v}_1, ..., \mathbf{v}_k\} \) is a linearly independent set.
(hint: first show this: Suppose \( T(\mathbf{w}) = \lambda \mathbf{w} \). Prove that \( (T - \mu I)^k(\mathbf{w}) = (\lambda - \mu)^k \mathbf{w} \) (10 points)

(6) Suppose that \( \mathbf{v}_1, ..., \mathbf{v}_k \in \mathbb{R}^n \) are orthonormal and that for every \( \mathbf{x} \in \mathbb{R}^n \) we have

\[ \| \mathbf{x} \|^2 = (\mathbf{x} \cdot \mathbf{v}_1)^2 + ... + (\mathbf{x} \cdot \mathbf{v}_k)^2. \]

Prove that \( k = n \) and deduce that \( \{\mathbf{v}_1, ..., \mathbf{v}_k\} \) is an orthonormal basis for \( \mathbb{R}^n \). (10 points)
Qualifying Exam (May 2008)

Answers, even if correct without relevant details will fetch very few points.

In the questions below: R denotes real numbers and Q denotes rational numbers.

1. (a) Show that if G is nonabelian, then the factor group $\frac{G}{Z(G)}$ is not cyclic. (Z(G) is the center of G)
   
   (b) Show that a nonabelian group G of order $pq$ where $p$ and $q$ are primes has a trivial center.

2. Show that a nonconstant polynomial in $C[x]$ has a zero in $C$ if and only if the following is true: Let $f_1(x), ..., f_r(x) \in C[x]$ and suppose that for every $\alpha \in C$ that is a zero of all $r$ of these polynomials is also a zero of a polynomial $g(x) \in C[x]$. Then some power of $g(x)$ is in the smallest ideal of $C[x]$ that contains the $r$ polynomials $f_1(x), ..., f_r(x)$.

3. What degree field extensions can we obtain by successively adjoining to a field $F$ a square root of an element of $F$ not a square in $F$, then square root of some nonsquare in this new field, and so on? Argue from this that a zero $x^{14} - 3x^2 + 12$ over $Q$ can never expressed as a rational function of square roots of rational functions of square roots, and so on, of elements of $Q$.

4. Let $A$ and $B$ be $2 \times 2$ matrices with integer entries such that $A, A + B, A + 2B, A + 3B$ and $A + 4B$ are all invertible matrices whose inverses have integer entries. Prove that $A + 5B$ is invertible and that its inverse has integer entries.
   
   (Hint: Look at the function $f(t) = \det(A + tB)$ for different values of $t$)

5. Let $A$ be an $n \times n$ matrix all of whose eigenvalues are real numbers. Prove that there is a basis for $R^n$ with respect to which the matrix for $A$ becomes upper triangular.

6. We say an $n \times n$ matrix $N$ with real entries is nilpotent if $N^r = 0$ for some positive integer $r$.
   
   (a) Show that 0 is the only eigenvalue of $N$.

   Prove directly, without using the Jordan form, the following:

   (b) Suppose $N^n = 0$ and $N^{n-1} \neq 0$. Prove that there is a basis \( \{v_1, ..., v_n\} \) for $R^n$ with respect to which the matrix for $N$ becomes

   \[
   \begin{bmatrix}
   0 & 1 & & \\
   0 & 1 & & \\
   & & \ddots & \\
   & & & 0 & 1 \\
   & & & & 0
   \end{bmatrix}
   \]

   (Hint: Might be helpful to show that the nullspace $N(N)$ is one dimensional)
Each question is given equal weight in the grading. \( \mathbb{Q} \) denotes the rational numbers, \( \mathbb{R} \) the reals, and \( \mathbb{C} \) the complex numbers. Fields should not be assumed to be the reals or complex numbers unless specifically stated. You should justify all assertions to receive full credit. Demonstrating an in-depth understanding of the material is the most important factor.

1. Let \( G \) be an abelian group. Let \( a, b \in G \). Prove that the order of \( ab \) is the least common multiple of the orders of \( a \) and \( b \) if the intersection of the subgroup \( A = \langle a \rangle \) and the subgroup \( B = \langle b \rangle \) is the identity.

(b) Prove that the alternating group \( A_5 \) has exactly five subgroups of order 4.

2. Let \( k \) be a field, and let \( f(x) \) and \( g(x) \) be irreducible polynomials in the polynomial ring \( k[x] \). Assume that the degrees of \( f(x) \) and \( g(x) \) are relatively prime. Let \( E \) be an extension field of \( k \), and assume that there is a root \( \alpha \) of \( g(x) \) with \( \alpha \in E \). Prove that \( f(x) \) is still irreducible, considered in the polynomial ring \( k(\alpha)[x] \).

3. Let \( R \) be a commutative integral domain containing a field \( k \). Assume that \( R \) is finite dimensional as a vector space over \( k \). Prove that \( R \) is a field.

4. Let

\[
A = \begin{pmatrix}
0 & 3 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 10 & 0 & -3 \\
1 & 0 & 1 & 0
\end{pmatrix}.
\]

State the Jordan canonical form theorem, and find the Jordan canonical form for the matrix \( A \), considered as a matrix with complex entries.

5. Let \( A \) be an \( n \) by \( n \) matrix over \( \mathbb{C} \). Assume that \( A \) has distinct eigenvalues. Let \( W \) be the subspace of complex \( n \) by \( n \) matrices which commute with \( A \). Prove that \( W \) has dimension at least \( n \).

6. Let \( A \) be an \( n \) by \( n \) real symmetric matrix such that \( T^k = I \) for some \( k \), where \( I \) is the identity matrix. Prove that \( T^2 = I \).
Each question is given equal weight in the grading. $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ denotes the rational numbers, $\mathbb{R}$ the reals, and $\mathbb{C}$ the complex numbers. Fields should not be assumed to be the reals or complex numbers unless specifically stated. You should justify all assertions to receive full credit. Although partial credit may be given, demonstrating an in-depth understanding of the material is the most important factor.

1. Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial. Let $\alpha \in \mathbb{C}$. Assume that both $\alpha$ and $3 \cdot \alpha$ are roots of $f(x)$. Prove that $f(0)$ is divisible by 3. (Hint: compare the minimal polynomials of $\alpha$ and $3 \cdot \alpha$ over $\mathbb{Q}$.)

2. Show that the degree of $\mathbb{Q}(\sqrt{5}, i)$ over $\mathbb{Q}$ is 4, where $i^2 = -1$. Find (with proof) an element $\alpha \in \mathbb{Q}(\sqrt{5}, i)$ such that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{5}, i)$.

3. A group $G$ is said to have exponent $k$ if $g^k = e$ for all $g \in G$. Let $n$ be a positive integer and let $\mathbb{Z}_n^*$ be the multiplicative group of units in the ring $\mathbb{Z}_n$. Find all values of $n$ such that $\mathbb{Z}_n^*$ has exponent 2.

4. Let $I$ be the ideal in $\mathbb{Z}[x]$ generated by the two polynomials $f = x^2 + 3x + 12$ and $g = x^2 + 3x + 82$. Find five maximal ideals containing $I$. Be sure to prove the ideals are maximal.

5. Let $T : V \to V$ be a linear transformation of an $n$-dimensional vector space over the complex numbers. Assume that the minimal polynomial of $T$ is $(x - 1)^n$. Find the Jordan canonical form of the operator $S = T^2$.

6. Let $A$ be an $n$ by $n$ matrix over the real numbers. Prove that $A$ is invertible if and only if the identity matrix $I$ is in the $\mathbb{R}$-span of the matrices $A, A^2, A^3, \ldots, A^n$. 

1
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1.

a) Find, up to isomorphism, all abelian groups of order 72 which do not contain an element of order 9. Clearly state any results you use.

b) Recall that an automorphism of a group \( G \) is a 1-1 and onto homomorphism of \( G \) to itself. The set of all automorphisms of \( G \) forms a group \( \text{Aut}(G) \). Let \( C \) be a cyclic group of order 21. Find (with proof) the order of \( \text{Aut}(C) \).

2. Let \( f(T) = T^3 + \omega T + \sqrt{3} \), where \( \omega \) is a primitive cube root of unity, let \( \alpha \) be a root of \( f \) in \( \mathbb{C} \), and set \( F = \mathbb{Q}(\alpha) \). Prove that \[ 4 \leq [F : \mathbb{Q}] \leq 12. \]

3. Let \( R = k[X, Y] \), where \( k \) is a field. Define a ring homomorphism \( f : R \to k[t] \) by \( f(p(X, Y)) = p(t^2, t^3) \).

a) Prove that the kernel of \( f \) is a prime ideal \( P \).

b) Prove that \( P \) is a principal ideal generated by \( X^3 - Y^2 \).

4.

a) Let \( v = (1, 1, 1, 1) \) and \( w = (5, 3, 3, 3) \), and let \( V \) be the subspace in \( \mathbb{R}^4 \) spanned by \( v \) and \( w \). Find an orthonormal basis for \( V \).

b) Give an example of matrices, none of which are the identity, which are 2 by 2 matrices over \( \mathbb{C} \), having each of the following properties: normal, orthogonal, and self-adjoint.

5. Let

\[
A = \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
-1 & 1 & 0
\end{pmatrix}
\]

and

\[
B = \begin{pmatrix}
0 & 1 & 4 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

Prove that \( A \) and \( B \) are similar.

6. Let \( T \) be a linear operator on a finite dimensional vector space \( V \) over \( \mathbb{C} \) whose characteristic polynomial is \( t^3(t - 2)^2 \). Find all possible Jordan normal forms for such an operator. What is the determinant of such an operator?
Algebra Qualifying Examination : January 2007

You must show all work to receive full credit. Throughout \( \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \) and \( \mathbb{C} \) denote respectively the integers, rational numbers, real numbers and complex numbers.

1. Consider the real valued functions \( f(x) := \frac{1}{x} \) and \( g(x) := \frac{x - 1}{x} \) defined on \( \mathbb{R} \setminus \{0, 1\} \). Let \( G \) be the group generated by \( f(x) \) and \( g(x) \) whose group law is composition of functions. Prove that \( G \) is isomorphic to the symmetric group \( S_3 \). (16 pts)

2. Let \( f(x) := x^3 - 3x - 3 \) belong to the polynomial ring \( \mathbb{Z}[x] \). For a prime number \( p \), let \( f_p(x) \) denote the polynomial with coefficients in \( \mathbb{Z}/p\mathbb{Z} \) obtained by reducing the coefficients of \( f(x) \) mod \( p \). For what \( p \) does \( f_p(x) \) have multiple roots? (16 pts)

3. Show that the polynomial rings \( \mathbb{Z}[x] \) and \( \mathbb{Q}[x] \) are not isomorphic as rings. (16 pts)

4. Let \( A \) be an \( n \times n \) matrix with entries in the field \( F \). Prove that the following numbers are the same: (a) the row rank of \( A \), (b) the column rank of \( A \) and (c) the determinantal rank of \( A \). Recall that the determinantal rank of \( A \) is the largest non-negative integer \( t \) such that determinantal \( (A') \neq 0 \), for some submatrix \( A' \) of \( A \) obtained by deleting \( n - t \) rows and columns of \( A \). (16 pts)

5. Let \( A := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \) and let \( T : \mathbb{C}^4 \to \mathbb{C}^4 \) be the linear transformation whose matrix with respect to the standard basis of \( \mathbb{C}^4 \) is \( A \). Equip \( \mathbb{C}^4 \) with the standard inner product.

(a) Find the Jordan canonical form of \( A \). (6 pts)

(b) Find an invertible \( 4 \times 4 \) matrix \( P \) such that \( P^{-1}AP \) is the matrix found in part (a). (6 pts)

(c) Does there exist a cyclic vector for \( T \)? If so, find one; if not, explain why not. (6 pts)

(d) Find a two-dimensional \( T \)-invariant subspace \( W \subseteq \mathbb{C}^4 \). (6 pts)

(e) Find an orthonormal basis for \( W \). (6 pts)

(f) Find the adjoint of \( T|_W \). (6 pts)
Each question is given equal weight in the grading. \( \mathbb{Q} \) denotes the rational numbers and \( \mathbb{C} \) denotes the complex numbers. Fields should not be assumed to be the reals or complex numbers unless specifically stated. You should justify all assertions to receive full credit. Demonstrating an in-depth understanding of the material is the most important factor.

1. Let \( G = S_n \) be the symmetric group on \( n \) elements, and let \( \sigma = (123...n) \) be an \( n \)-cycle. Let \( K \) be the cyclic subgroup generated by \( \sigma \). Prove that the order of the normalizer of \( K \), i.e., the order of the subgroup \( H = \{ x \in S_n \mid x^{-1} \sigma x \in K \} \), is exactly \( n \cdot \phi(n) \), where \( \phi(n) \) is the Euler \( \phi \) function. (Recall that \( \phi(n) \) is the number of positive integers less than \( n \) and relatively prime to \( n \).)

2. Let \( \alpha \in \mathbb{C} \) be a primitive 9th root of unity. Find the degree of the minimal polynomial of \( \beta = \alpha + \alpha^{-1} \) over \( \mathbb{Q} \) (with proof).

3. Let \( k \) be the finite field with 3 elements. Let \( f(x) \) be an irreducible polynomial of degree \( n \) in \( k[x] \). Prove that \( k[x]/(f(x)) \) is a field with \( 3^n \) elements. Prove that there are infinitely many non-associate irreducible polynomials in \( k[x] \). (Two polynomials \( f, g \) are associates if \( (f) = (g) \).)

4. Let \( V \) be a finite-dimensional inner product space over the complex numbers. Recall a linear operator \( T \) is said to be self-adjoint if \( T = T^* \), where \( T^* \) is the adjoint of \( T \). If \( T \) is self-adjoint, prove that \( I + iT \) is invertible. (Where \( i = \sqrt{-1} \).)

5. Let \( A \) be a complex 6 by 6 matrix which has exactly two eigenvalues, 0 and 1, and has rank 4. Give all possible minimal polynomials of such a matrix, and give an example of a matrix with each possible minimal polynomial.

6. Let \( T \) be a linear operator on a finite dimensional vector space \( V \) over a field \( F \). Prove that

\[
\text{rank}(T^3) + \text{rank}(T) \geq 2 \cdot \text{rank}(T^2).
\]
Algebra Qualifying Examination

January 9, 2006

In what follows \( F \) will always denote a field.

1. Let \( G \) be a finite group. Give a complete proof (without quoting any theorem) that there is an integer \( n \) such that \( a^n = e \) for all \( a \in G \).

2. Let \( K \trianglelefteq N \trianglelefteq G \) be groups and subgroups. Assume that \( K \) is normal in \( N \), \( N \) is normal in \( G \) and \( K \) is normal in \( G \). Prove that there is an isomorphism

\[
\frac{G/K}{N/K} \cong \frac{G}{N}.
\]

3. Let \( R = \mathbb{C}([0, 1]) \) be the ring of all continuous real valued functions on the interval \([0, 1]\). Let \( 0 < p < 1 \) and

\[
I_p = \{ f \in R : f(p) = 0 \}.
\]

Prove that \( I_p \) is a maximal ideal in \( R \).

4. Let \( F \) be a field and \( R = F[X] \) be the polynomial ring over \( F \). Prove that \( R \) is a PID.

5. Let \( R \) be a commutative ring and \( I, J \) be two ideals such that \( I + J = R \). Prove that the map

\[
f : R/(I \cap J) \to R/I \oplus R/J \text{ given by } f(x + I \cap J) = (x + I, x + J)
\]

is an isomorphism.
6. Let $\mathbb{F}$ be a field and $V = \mathbb{F}^n$. Think of the elements of $V$ as row vectors. Let $v_1, v_2, \ldots, v_n \in V$ and let

$$A = \begin{pmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{pmatrix}$$

Prove that $v_1, v_2, \ldots, v_n$ forms a basis if and only if $A$ invertible.

7. Let $V, W$ be two finite dimensional vector spaces over a field $\mathbb{F}$ and let $T : V \to W$ be a linear transformation. Let $\mathcal{N}$ be the null space of $T$ and $\mathcal{R} = T(V)$ be the range of $T$. Prove that

$$\dim(\mathcal{N}) + \dim(\mathcal{R}) = \dim(V).$$

8. Suppose $V$ is a finite dimensional vector space over a field $\mathbb{F}$. Suppose $T : V \to V$ is a linear operator on $V$. Suppose $c_1, c_2 \in \mathbb{F}$ be two distinct eigen values of $T$. Let

$$\mathcal{N}(c_i) = \{v \in V : T(v) = c_i v\}$$

be the eigen spaces of $c_i$. Prove that $\mathcal{N}(c_1) \cap \mathcal{N}(c_2) = \{0\}$.

9. Suppose $V$ is a vector space over $\mathbb{F}$ with $\dim V = n$. Suppose $T \in L(V, V)$ is an operator on $V$. Suppose $W_0$ is an $T$-invariant subspace of $V$ and $v \in V \setminus W_0$. Write $W_1 = W_0 + \mathbb{F}[T]v$ and

$$I = \{f \in \mathbb{F}[X] : f(T)v \in W_0\}.$$

Let $f$ be the minimal monic polynomial of $I$. Prove that $\dim W_1 = \dim W_0 + \text{degree}(f)$.

10. Let $V$ be an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For $x, y \in V$, prove the Cauchy-Schwartz inequality that

$$| (x, y) | \leq \| x \| \| y \|$$

and that the equality holds if and only if

$$y = \frac{(y, x)}{\| x \|^2} x.$$
ALGEBRA QUALIFYING EXAMINATION : AUGUST 2005

You must justify your answers to receive full credit.

1. Recall that $A_4$ denotes the alternating group on 4 letters.
   (a) What are the possible indices of subgroups of $A_4$? Explain. (5 points)
   (b) For each response in (a), either give an example of a subgroup with that index or explain why no such subgroup exists. (15 points)

2. Let $f(X)$ be an irreducible polynomial of degree 35 with coefficients in $\mathbb{Q}$. Let $\alpha \in \mathbb{C}$ be a root of $f(X)$.
   (a) Show that $\alpha^{29} \notin \mathbb{Q}$. (5 points)
   (b) Show that $\mathbb{Q}(\alpha^4) = \mathbb{Q}(\alpha)$. (10 points)

3. Let $R$ be a commutative ring and $R[X]$ the polynomial ring with coefficients in $R$. For an ideal $J \subseteq R$, write $J[X]$ for the set of polynomials with coefficients in $J$.
   (a) If $J$ is a prime ideal prove that $J[X]$ is a prime ideal. (10 points)
   (b) Prove or disprove a similar statement for maximal ideals. (10 points)

4. Let $S$ be the subspace of $\mathbb{R}^4$ spanned by the vectors $(1,0,0,1)$ and $(0,1,-1,1)$. Let $T$ be the orthogonal complement of $S$ in $\mathbb{R}^4$. Find the projection of the vector $(1,1,1,1)$ onto:
   (a) $S$ (b) $T$. (15 points)

5. Let $A := \begin{pmatrix} -5 & 0 & 0 \\ 1 & -7 & -2 \\ -1 & 2 & -3 \end{pmatrix}$ represent a linear operator $T$ on $\mathbb{C}^3$. Determine whether or not $T$ has a cyclic vector. (15 points)

6. Let $A$ and $B$ be $6 \times 6$ nilpotent matrices over a field $F$. Suppose that $A$ and $B$ have the same minimal polynomial and the same nullity. Prove that $A$ and $B$ are similar. (15 points)
You must show all work to receive full credit. Each problem is worth 15 points.

1. Let $G$ be a finite group such that $|G|$ is square-free. Assume that $x \cdot y = y \cdot x$ for all $x, y \in G$ such that the order of $x$ is relatively prime to the order of $y$. Use Cauchy's theorem to prove that $G$ is cyclic.

2. Let $\alpha$ denote the complex number $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot i$. Find, with proof, the minimal polynomial for $\alpha$ over $\mathbb{Q}$ and use it to write $\alpha^{105}$ as a polynomial in $\alpha$ of least degree.

3. Fix a prime number $p$ and let $R$ denote the subring of $\mathbb{Q}$ consisting of those fractions whose denominator is not divisible by $p$.

   (a) Show that every proper ideal of $R$ has the form $p^k R$, for some $k \geq 1$. Conclude that $R$ is a PID with only one maximal ideal.

   (b) Show that $R$ modulo its unique maximal ideal is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

   (c) Let $X$ be an indeterminate and $R[X]$ the polynomial ring with coefficients in $R$. Show that the principal ideal generated by $pX - 1$ is a maximal ideal.

4. Let $A$ be a $3 \times 3$ matrix with entries in $\mathbb{R}$. Prove that if $A$ is not similar over $\mathbb{R}$ to a triangular matrix, then $A$ is similar over $\mathbb{C}$ to a diagonal matrix.

5. Describe all rational canonical forms for $11 \times 11$ matrices over $\mathbb{Q}$ with minimal polynomial $(X - 5)^3(X^2 + 5X + 3)^2$.

6. Let $T$ be a linear operator on a finite dimensional inner product space $V$ over $\mathbb{C}$. Prove that $T$ is self adjoint if and only if $\langle T(v), v \rangle$ is a real number for every $v \in V$. 
ALGEBRA QUALIFYING EXAMINATION : JANUARY 2005

You must show all work to receive full credit.

1. Let $G$ be a finite abelian group. Prove that $G$ is cyclic if and only if for each positive integer $n$, the set \( \{ x \in G \mid x^n = e \} \) has at most $n$ elements. (16 points)

2. Let $f(X) = X^3 + 9X + 15 \in \mathbb{Q}[X]$.
   (a) Let $\alpha$ be a real root of $f(X)$ and let $F$ denote the set of all expressions of the form $\lambda_0 + \lambda_1 \cdot \alpha + \lambda_2 \cdot \alpha^2$, where each $\lambda_i \in \mathbb{Q}$. Give a rigorous proof that $F$ is a subfield of $\mathbb{R}$ containing $\mathbb{Q}$. (10 points)
   (b) Find the multiplicative inverse of $\alpha^2$ in $F$. (8 points)

3. (a) Give an example of a commutative integral domain containing a non-zero prime ideal that is not a maximal ideal. (8 points)
   (b) Let $R$ be a principal ideal domain and $J \subseteq R$ an ideal. Prove that the following are equivalent (10 points):
      
      (1) $J = aR$, for $a \in R$ an irreducible element
      (2) $J = aR$, for $a \in R$, a prime element
      (3) $J$ is a prime ideal
      (4) $J$ is a maximal ideal

Recall that a non-zero element $a$ in an integral domain is prime if $a|c \cdot d$, then $x|c$ or $x|d$ and $a$ is an irreducible element if $a = c \cdot d$ implies $c$ or $d$ is a unit.

4. Let $V$ be a finite-dimensional vector space over the field $F$ and $T$ be a linear operator on $V$. Let $c$ be a scalar and suppose there exists a nonzero vector $v$ in $V$ such that $Tv = cv$. Prove that there exists a nonzero linear functional $f$ on $V$ such that $f \circ T = cf$. (16 points)

5. Prove the following or give a counter-example. If $E_1$ and $E_2$ are linear projections on a finite-dimensional vector space $V$ such that $\text{range}(E_1) = \text{range}(E_2)$, then $\ker(E_1) = \ker(E_2)$. (16 points)

6. Let $V$ be the space of complex $n \times n$ matrices with the inner product $(A|B) := \text{tr}(AB^*)$. For each $M$ in $V$, let $T_M$ be the linear operator defined by $T_M(A) = MA$. Show that $T_M$ is unitary if and only if $M$ is unitary. (16 points)
Algebra Qualifying Examination

1. a) Show that a subgroup of a cyclic group is cyclic.
   
b) Is the following “converse” of the above statement true: If a group $G$ is such that every proper subgroup is cyclic, then $G$ is cyclic.

2. a) Show that any group homomorphism $\Phi : G \rightarrow G'$, where the order of $G$ is prime must either be the trivial homomorphism or a one-to-one map.
   
b) Does there exist a surjective homomorphism from a group of order 14 to a group of order 6? Give reasons.
   
c) How many group homomorphisms are there of $\mathbb{Z}$ onto $\mathbb{Z}$?

3. a) Show that every finite integral domain is a field. Deduce that if $R$ is a finite commutative ring with identity, then every prime ideal is maximal.
   
b) Show that $X^3 + 3X^2 - 8$ is irreducible over the field of rational numbers $\mathbb{Q}$.
   
c) Is $\mathbb{Q}[x] / (x^2 - 5x + 6)$ a field? Why?

4. a) Show that the field of real numbers $\mathbb{R}$ is not a finite field extension of $\mathbb{Q}$.
   
b) Prove that the algebraic closure of $\mathbb{Q}$ in $C$ is not a finite extension of $\mathbb{Q}$.
   
   (Hint: use Eisenstein criterion to show that $\mathbb{Q}$ has finite extensions in $C$ of arbitrarily large degrees.)

5. Let $V$ and $W$ be vector spaces over the field $F$ and let $T$ be a linear transformation from $V$ into $W$. Suppose that $V$ is finite dimensional. Prove: $\text{rank } (T) + \text{nullity } (T) = \dim V$.

6. Let $D$ be the differentiation operator on the space of polynomials in $F[t]$ of degree less than or equal to $n$.
   
a) Find the matrix of $T$ in the ordered basis $\{1, t, t^2, t^3, \ldots, t^n\}$.
   
b) Find the Jordan form of the matrix in part (a). Justify your answer.

7. Let $A$ be an $n \times n$ nilpotent matrix. Prove $A^n = 0$. 
1. (17 pts) There are five groups listed below. Tell which ones are isomorphic to which of the others, and justify your answer. If two are not isomorphic, explain why not. $C_n$ is the cyclic group of order $n$, and $S_n$ is the symmetric group on $n$ elements.

a) $C_2 \times C_{18}$
b) $C_6 \times S_3$
c) $S_3 \times S_3$
d) $C_2 \times C_2 \times C_9$
e) $C_6 \times C_6$

2. (6 pts each) The following three questions concern the ring $R = \mathbb{Q}[X,Y]/I$, where $I$ is the ideal generated by the square of the ideal $(X,Y)$, i.e., $I = (X,Y)^2$.

a) Describe (with proof) all the ideals of $R$.
b) Describe the units of $R$, that is the elements which have multiplicative inverses.
c) Prove there is a nontrivial ring homomorphism from $R$ to the complex numbers.

3. (15 pts) Let $\mathbb{Q}$ be the field of rational numbers, and let $R = \mathbb{Q}[X]$ be a polynomial ring over $\mathbb{Q}$. Let $f(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_0 \in R$. Let $A$ be the companion matrix of $f$, namely

$$A = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & 0 & \ldots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -a_{n-1} \end{pmatrix}$$

Prove that $f(X)$ is both the minimal polynomial and the characteristic polynomial of $A$.

4. (15 pts) Find a 6 by 6 matrix $A$ over the complex numbers such that $A^4 = 0$ but $A^3 \neq 0$. Let $B$ be another 6 by 6 matrix satisfying these two conditions. Must $A$ be similar to $B$ (explain your answer)?

5. (15 pts) Let $\alpha$ be a root in the complex numbers of the monic polynomial $f(x) = x^3 + x + 1$. Set $K = \mathbb{Q}(\alpha)$. Prove or disprove the following statement: $\sqrt{-1} \in K$.

6. (10 pts each)

a) Let $V$ be an inner product space. Define what it means to say an operator $U$ on $V$ is normal.

b) Let $T$ be an operator on the inner product space $V$. Prove that $T$ preserves the inner product if and only if for every $v \in V$, $\|Tv\| = \|v\|$.
ALGEBRA QUALIFYING EXAMINATION: MAY 2004

YOU MUST JUSTIFY YOUR ANSWERS TO RECEIVE FULL CREDIT.

1. State the class equation. Verify the class equation for $S_4$ listing its elements and grouping them according to the class equation.

2. A commutative integral domain $V$ is said to be a valuation domain if for every non-zero $a, b \in V$, either $a | b$ or $b | a$.
   (a) Show that the set of non-units form an ideal, $M$.
   (b) Show that $M$ is the unique maximal ideal of $V$.
   (c) Let $p$ be a prime number and $V$ be the set of rational numbers $\frac{a}{b}$ in lowest terms such that $p$ does not divide $b$. Show that $V$ is a valuation domain.

3. Let $f(X) = X^3 + 2X^2 + X + 3$ be a polynomial with coefficients in the finite field $Z_5$. Let $\alpha$ be a root of $f(X)$ and let $F$ be the field $Z_5(\alpha)$.
   (a) Show $f(X)$ is irreducible over $Z_5$.
   (b) Describe the field $F$ and determine the number of elements in $F$.
   (c) Let $a, b \in F$, and let $g(X) = X^2 + aX + b$ be an irreducible polynomial in $F[X]$. Let $\gamma$ be a root of $g(X)$, and set $K = F(\gamma)$. How many elements does $K$ have?

4. (a) Let $A$ and $B$ be similar $n \times n$ matrices over the field $F$. How do $tr(A)$ and $det(A)$ compare with $tr(B)$ and $det(B)$? Justify your answer.
   (b) Consider the matrices

   $$A = \begin{pmatrix} 1 & 1 & -4 & 9 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

   Are they similar over $\mathbb{C}$? Over $\mathbb{R}$?
5. Let $V$ be the vector space of real polynomials of the form

$$f(X) = a_8 X^8 + a_7 X^7 + a_6 X^6 + a_5 X^5$$

and $W$ the subspace of polynomials $f \in V$ satisfying $a_8 + \cdots + a_5 = 0$.

(a) Calculate $\dim(W)$.

(b) Let $\langle , \rangle$ be the inner product on $W$ given by $\langle f, g \rangle = \int_0^1 f \cdot g \, dx$, for all $f, g \in W$. Find an orthonormal basis for $W$.

6. Let $V$ be an $n$-dimensional vector space over $\mathbb{R}$ and $T : V \to V$ a linear operator. Show that $T$ has a non-zero eigenvector in each of the following cases: (a) $n$ is odd (b) $n > 0$ and $T$ is self-adjoint.
Algebra Qualifying Examination : January 2004

Throughout, \( \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \) denote, respectively, the fields of rational, real and complex numbers and \( \mathbb{Z} \) denotes the ring of integers.

1. Let \( P \) be a Sylow 2-subgroup of \( S_5 \), the symmetric group on 5 elements.
   
   a) What is the order of \( P \) ?
   
   b) What are the possible orders of the elements of \( P \), and how many elements have each given order ?

2. Let \( \mathbb{C}^* \) be the multiplicative group of non-zero elements in \( \mathbb{C} \).
   
   a) Let \( x \) in \( \mathbb{C}^* \) be an element of finite order. Show that \( x \) is algebraic over \( \mathbb{Q} \).
   
   b) Assume \( n \) divides \( m \). Show that \( \mathbb{Z}_n \times \mathbb{Z}_m \) cannot be isomorphic to a subgroup of \( \mathbb{C}^* \).
   
   c) For each positive integer \( n \), find a cyclic subgroup of \( \mathbb{C}^* \) of order \( n \).

3. Let \( A \) be the ring of continuous real valued functions on \([0,1]\) (with point-wise addition and multiplication). For \( t \) in \([0,1]\), set \( M_t = \{ f \in A \mid f(t) = 0 \} \). Prove that \( M_t \) is a maximal ideal (including details that \( M_t \) is an ideal).

4. Let \( F \) be a field. For a positive integer \( n \), let \( I_n \) denote the \( n \times n \) identity matrix over \( F \).
   
   a) If \( F = \mathbb{R} \), prove that there do not exist \( n \times n \) matrices \( A, B \) over \( F \) satisfying \( AB - BA = I_n \).
   
   b) If \( F = \mathbb{Z}_2 \), show that there exist \( 2 \times 2 \) matrices \( A, B \) over \( F \) satisfying \( AB - BA = I_2 \).

5. Let \( V \) denote the vector space of \( 2 \times 2 \) matrices over \( \mathbb{C} \) and set \( B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Define a linear operator \( T : V \to V \) by \( T(A) = AB - BA \), for all \( A \) in \( V \). Find the Jordan canonical form of \( T \).

6. Let \( V \) be a finite dimensional inner product space over \( \mathbb{C} \) and \( T \) a linear operator on \( V \).
   
   a) Show that the adjoint of \( T \) is unique.
   
   b) Now assume that \( V \) is the vector space of \( n \times n \) complex matrices with inner product given by \( \langle A, B \rangle := \text{tr}(B^*A) \), where \( B^* \) denotes conjugate transpose. Fix an \( n \times n \) matrix \( L \) and let \( T : V \to V \) be the linear operator on \( V \) given by \( T(A) = LA \), for all \( A \) in \( V \). Show that \( T^* \) is left multiplication by \( L^* \), i.e., \( T^*(A) = L^*A \), for all \( A \) in \( V \).
1. Let $G$ be a finite group and recall that elements $x$ and $y$ in $G$ are said to be *conjugate* if there exists $g \in G$ such that $y = g^{-1}xg$.
   
   (a) Show that if $G$ has only two conjugacy classes, then $|G| = 2$. (7 pts)
   
   (b) Find the number of conjugacy classes in the symmetric group $S_4$. (8 pts)

2. (a) Let $f(x)$ be a degree four monic polynomial with integer coefficients. Answer the following questions true or false (no justification required):
   
   (i) If $f(x)$ factors non-trivially over $\mathbb{Q}$, then it factors non-trivially over $\mathbb{Z}$. (3 pts)
   
   (ii) $f(x)$ factors as a product of quadratic polynomials over $\mathbb{R}$. (3 pts)
   
   (iii) $f(x)$ factors as a product of linear polynomials over $\mathbb{C}$. (3 pts)
   
   (b) Prove that $x^4 + x + 1$ is irreducible over $\mathbb{Q}$. (11 pts)

3. Let $R := \mathbb{Z}[x]$ be the ring of polynomials with integer coefficients.
   
   (a) Describe the ideal $5R$ and show that it is a prime ideal. (5 pts)
   
   (b) Let $M := (5, x^2 + 2)R$. Show that $M$ is a maximal ideal. (5 pts)
   
   (c) How many elements are in the field $R/M$? (5 pts)

4. Find a $3 \times 3$ matrix $A$ over $\mathbb{R}$ such that $A \neq I_3$, yet $A^3 = I_3$. (15 pts)

5. Fix a $5 \times 5$ matrix $A$ over $\mathbb{C}$. Define $I_A$ to be the ideal of polynomials $f(x)$ in $\mathbb{C}[x]$ such that $f(A) = 0$. Suppose $I_A$ is generated by $(x - 1)^3(x - 2)^2$ and $(x - 1)^2(x - 2)^3$. What are the possible Jordan canonical forms for $A$? (Warning: the given generating set for $I_A$ is not necessarily a minimal generating set.) (15 pts)

6. Consider the complex vector space $\mathbb{C}^3$ with the usual inner product. For each $v, w \in \mathbb{C}^3$, let $T_{v,w}$ be the map on $\mathbb{C}^3$ defined by $T_{v,w}(x) = \langle x, w \rangle v$. Show that:
   
   (a) $T_{v,w}$ is a linear operator on $\mathbb{C}^3$. (5 pts)
   
   (b) $(T_{v,w})^* = T_{w,v}$. (5 pts)
   
   (c) $\text{trace}(T_{v,w}) = \langle v, w \rangle$. (5 pts)
   
   (d) When is $T_{v,w}$ self-adjoint? (5 pts)
1. Let $G$ be a finite group. The *commutator subgroup* of $G$, denoted $[G, G]$, is the set of all finite products of elements of the form $g^{-1}h^{-1}gh$, where $g$ and $h$ range over all elements of $G$.

a) (8 pts) Prove that $[G, G]$ is a subgroup of $G$, and prove that it is a normal subgroup.

b) (7 pts) Let $H$ be a normal subgroup of $G$. Prove that $G/H$ is abelian if and only if $[G, G] \subseteq H$.

2. (10 pts each) The following two questions concern the ring $R = \mathbb{Z}[X]/I$, where $I$ is the ideal generated by the elements $X^2 + 1$ and 6.

a) Describe (with proof) all the maximal ideals of $R$.

b) Prove that $R$ is a finite ring, and find the number of elements of $R$.

3. (15 pts) Let $\mathbb{Q}$ be the field of rational numbers, and let $R = \mathbb{Q}[X]$ be a polynomial ring over $\mathbb{Q}$. Let $f(X) \in R$ be a monic irreducible polynomial in $R$ of degree $n$, and let $\alpha$ be a complex root of $f(X)$. Set $K = \{a_0 + a_1\alpha + \ldots + a_k\alpha^k + \ldots + a_{n-1}\alpha^{n-1} | a_0, \ldots, a_{n-1} \in \mathbb{Q}\}$. Prove that $K$ is a subfield of the complex numbers.

4. (15 pts) Let $A$ be a $4 \times 4$ matrix with entries in the complex numbers. Assume that $A^3 - 3A^2 + 2A = 0$. Using Jordan normal form, classify up to similarity all possible such $A$.

5. (15 pts) Let $V$ be an finite dimensional vector space over a field $F$, and let $T$ be a linear operator on $V$. Prove that $V = \text{Ker}(T) \oplus \text{Image}(T)$ if and only if $\text{Ker}(T) = \text{Ker}(T^2)$. Recall that $\text{Ker}(T) = \{v \in V | T(v) = 0\}$ and $\text{Image}(T) = \{T(v) | v \in V\}$.

6. (10 pts each) Let $V$ and $W$ be finite dimensional inner product spaces over the complex numbers.

a) Suppose that $T : V \rightarrow W$ is a linear operator and let $T^*$ be its adjoint, i.e., $T^* : W \rightarrow V$ satisfies $\langle T(v), w \rangle = \langle v, T^*(w) \rangle$ for all $v \in V$ and $w \in W$. Prove that $T$ is injective if and only if $T^*$ is surjective.

b) Suppose that $T : V \rightarrow V$ is a linear operator on $V$ which is self-adjoint. Prove that $T^3 = I$ implies that $T = I$, where $I$ is the identity operator.
1. Determine whether the polynomial in \( \mathbb{Z}[x] \) satisfies the Eisenstein Criteria for irreducibility over \( \mathbb{Q} \)

\[ 2x^2 - 25x^3 + 10x^2 - 30 \]

2. Show that a group with at least two elements but with no proper nontrivial subgroups must be finite and of prime order.

3. Let \( G \) be a group of order \( pq \), \( p \), \( q \) prime numbers. Show that every proper subgroup of \( G \) is cyclic.

4. Show that \( 1 \) and \( p - 1 \) are the only elements of field \( \mathbb{Z}_p \) that are their own multiplicative inverse.

5. Let \( E \) be an extension field of a finite field \( F \), where \( F \) has \( q \) elements. Let \( \alpha \in E \) be algebraic over \( F \) of degree \( n \). Prove that \( F(\alpha) \) has \( q^n \) elements.

6. a) Show that there exists an irreducible polynomial of degree 3 in \( \mathbb{Z}_3[x] \).

b) Use (5) and (a) to show that there exists a field of 27 elements.

c) Does there exist a field with 28 elements? Explain.

7. Let \( A = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 -3 \end{bmatrix} \)

   a) Find the characteristic polynomial of \( A \).

   b) Find the minimal polynomial of \( A \).

   c) Find the Jordan form of \( A \).

8. Let \( n \) be a positive integer. Let \( V \) be the real space of all polynomials over \( \mathbb{R} \) of degree at most \( n \). Let \( T : V \rightarrow V \) be the linear transformation defined by

\[ T(f(t)) = tf'(t) - (n + 1)f(t), \text{ where } f'(t) \text{ represents the derivative of } f(t). \]

Show that \( \det(T) \neq 0 \).

9. Let \( W \) be a subspace of a finite dimensional vector space \( V \). Prove: \( \dim W + \dim W^\perp = \dim V \).
1. (Group Theory) (10 pts each)
   a) Find the maximal order of an element in the symmetric group $S_{10}$. Explain what results you use to find the maximal order.
   b) Let $G$ be a group. Prove that a subgroup $H$ is normal in $G$ if and only if $H$ is the kernel of a homomorphism from $G$ to another group.

2. (Rings) (10 pts) Recall that the characteristic of a commutative ring $R$ with identity $1$ is the least integer $n$ such that $n \cdot 1 = 0$. Find (with proof) the characteristic of the ring $R = \mathbb{Z}_6 \times \mathbb{Z}_2 \times \mathbb{Z}_{15}$.

3. (Integers, Fields and Polynomials) (10 pts each)
   a) Prove that $x^3 + 3x^2 - 8$ is irreducible over the rational numbers.
   b) Let $K \subseteq L$ be fields. Set $F = \{ \alpha \in L | \alpha \text{ is algebraic over } K \}$. Prove that $F$ is a field.
   c) Prove that for all integers $n$, $n^{37} - n$ is divisible by 114.

4. (Matrices) (15 pts) Let
   \[ A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{pmatrix}. \]

   Find the characteristic polynomial, the minimal polynomial, and the Jordan normal form of $A$.

5. (Linear Operators and Vector Spaces)
   a) (8 pts) Let $V$ be an $n$-dimensional vector space over a field $F$, with $n \geq 2$. Suppose that $N$ is a linear operator on $V$ such that $N^{n-1} \neq 0$ but $N^n = 0$. Prove that there is not an operator $T$ on $V$ such that $T^2 = N$.
   b) (7 pts) Let $V$ be a finite dimensional vector space, and suppose that $T$ and $S$ are two linear operators on $V$ which commute. Let $\lambda$ be an eigenvalue of $T$, and set $W = \{ v \in V | Tv = \lambda v \}$. Prove that $S(W) \subseteq W$.

6. (Inner Product Spaces) (10 pts.) Let $T$ be a self-adjoint linear operator on a finite dimensional complex inner product space. Let $I$ be the identity operator. Prove that $S = I + iT$ is invertible. (Hint: consider $(Sv|Sv)$.)
Qualifying Exam

Algebra

August, 2001

There are two pages to this exam.

1. (Group Theory) Let $G$ be a simple group of order 168 (such a group does exist). For each of the following numbers, say whether or not $G$ has a subgroup of that order. Give reasons. 'Maybe' is not an accepted answer.

a) (4 pts) Order 4.
b) (3 pts) Order 8.
c) (3 pts) Order 11.
d) (3 pts) Order 21.
e) (3 pts) Order 42.
f) (4 pts) Order 84.

2. (Fields and Polynomials) (10 pts) Let $\alpha = \sqrt{3}$ in the complex numbers and let $\beta$ be a cube root of 7. Suppose that $a, b, c, d, e, f$ are rational numbers and

$$a + b\alpha + c\beta + d\beta^2 + e\alpha\beta + f\alpha\beta^2 = 0.$$

Prove that $a = b = c = d = e = f = 0$.

3. (Group Theory)

a) (10 pts) Let $G$ be a group of order 105. Prove that $G$ is not simple.
b) (10 pts) List all abelian groups (up to isomorphism) of order 225.

4. (Inner Product Spaces) Let $V$ be a finite-dimensional inner product space over the complex numbers.

a) (5 pts) Let $T$ be a linear operator on $V$. Suppose that for all $v \in V$, $(Tv|v) = 0$. Prove that the only possible characteristic value of $T$ is 0.
b) (10 pts) Let $T$ be a linear operator on $V$. Set $U = \text{Ker}(T)$ and set $W = \text{Im}(T^*)$. Prove that $U = W^\perp$.

5. (Linear Operators) (20 pts) Let

$$A = \begin{pmatrix}
3 & -4 & -4 \\
-1 & 3 & 2 \\
2 & -4 & -3
\end{pmatrix}.$$

Find the characteristic polynomial, the minimal polynomial, the Jordan normal form and the rational canonical form of $A$. 
6. (Vector Spaces and Linear Functionals)
   a) (7 pts) Let $V$ be the vector space of all $n$ by $n$ matrices over a field $F$. Fix an $n$ by $n$ matrix $B$, and define a linear operator $T : V \rightarrow V$ by $T(A) = AB - BA$. Prove that $\det(T) = 0$.
   b) (8 pts) Let $V$ be a finite dimensional vector space over a field $F$. Prove that $V$ is isomorphic to its double dual $V^{**}$. 

1. (Group Theory) Give examples of each of the following:
   a) (2 pts) A nonabelian finite group with a normal subgroup of index 2.
   b) (3 pts) Two finite nonabelian, non-isomorphic groups of the same order.
   c) (2 pts) A finite nonabelian simple group.
   d) (3 pts) A finite abelian group which is not isomorphic to the product of two cyclic groups.

2. (Fields and Polynomials)
   a) (10 pts) Let $\alpha$ be a nonzero complex number. Suppose that $\alpha$ is the root of a monic polynomial of degree $n$ with rational coefficients. Prove that $\beta = \alpha^2 + \alpha^{-1}$ is also the root of a monic polynomial with rational coefficients of degree at most $n$.
   b) (10 pts) Prove that $x^6 + x^3 + 1$ is irreducible over the rational numbers.

3. (Group Theory)
   a) (5 pts) State the Class Equation for a finite group $G$.
   b) (5 pts) Let $G$ be a group of order 81, and suppose that the center of $G$ has order 9. How many conjugacy classes does $G$ have? (Justify your answer.)
   c) (10 pts) Let $G$ be a group of order $2^4 \cdot 3 \cdot 7^2$. Prove that $G$ is not simple.

4. (Inner Product Spaces) Let $V$ be a finite dimensional inner product space over the complex numbers.
   a) (4 pts) If $W$ is a subspace of $V$, prove that $W^\perp \oplus W = V$.
   b) (3 pts) Let $T : V \to V$ be a linear operator. If $(Tv|w) = 0$ for all $v, w \in V$, prove that $T$ is the zero operator.
   b) (8 pts) Let $T : V \to V$ be a self-adjoint linear operator. Prove that all the characteristic values of $T$ are real, and prove that characteristic vectors associated to distinct characteristic values are orthogonal.
5. (Linear Operators)

a) (10 pts) Let

\[ A = \begin{pmatrix}
0 & 0 & 2 & 1 \\
0 & 0 & -3 & 2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \]

Find the rational normal form and the Jordan canonical form for \( A \).

b) (10 pts) Classify up to similarity all 4 by 4 complex matrices \( A \) which satisfy \( A^3 = A^2 \).

6. (15 pts) (Vector Spaces and Linear Functionals) Let \( V \) be a finite dimensional vector space over a field \( F \). Assume \( \dim(V) = n \). Let \( f_1, ..., f_n \in V^* \). Define a linear operator

\[ T : V \rightarrow F^n \]

by \( T(v) = (f_1(v), ..., f_n(v)) \) for \( v \in V \). Prove that \( T \) is an isomorphism if and only if \( f_1, ..., f_n \) are linearly independent.
1. If $G$ is a group and $H$ is a subgroup of $G$, recall that
$N_G(H) = \{x \in G \mid x^{-1}Hx \subseteq H\}$.

   a) (5 pts) State the three theorems of Sylow.
   b) (5 pts) Let $G$ be a finite group, and let $p$ be a prime number dividing the order of $G$. Let $P$ be a $p$-Sylow subgroup of $G$. Prove that $P$ is the unique $p$-Sylow subgroup of $N_G(P)$.
   c) (10 pts) Let $G$ be a finite group, and let $p$ be a prime number dividing the order of $G$. Let $P$ be a $p$-Sylow subgroup of $G$. Prove that $N_G(N_G(P)) = N_G(P)$.

2.
   a) (5 pts) State Gauss's Lemma.
   b) (10 pts) Let $\alpha$ be a complex number. Assume that there is a monic polynomial $f(X)$ with integer coefficients such that $f(\alpha) = 0$. Prove that the minimal polynomial of $\alpha$ over the rational numbers also has integer coefficients. (Recall that by definition, the minimal polynomial is monic.)

3. Let $\alpha$ be a primitive 8th root of unity in the complex numbers.
   a) (5 pts) Find the minimal polynomial for $\alpha$ over the rational numbers (with proof).
   b) (5 pts) Write the inverse of $\alpha + 2$ as a polynomial in $\alpha$ with coefficients in the rational numbers.
   c) (5 pts) Let $R = \mathbb{Z}[\alpha] = \{f(\alpha) \mid f(X) \in \mathbb{Z}[X]\}$. What is the cardinality of $R/(5R)$?
   d) (5 pts) Describe all the maximal ideals in the ring $R/(5R)$.

4. (15 pts) Let $V$ be a finite dimensional vector space over a field $F$, and let $T : V \rightarrow V$ be a linear transformation. Write $p(X)$ for the minimal polynomial of $T$ and $q(X)$ for the characteristic polynomial of $T$. Prove that for all $\lambda \in F$, $q(\lambda) = 0$ if and only if $p(\lambda) = 0$.

5. (15 pts) Let $V$ be the subspace of $\mathbb{R}^4$ generated by the vectors $[1, 0, 1, 0], [1, 0, 1, 1]$, and $[-1, 0, 1, 0]$. Find an orthonormal basis for $V$.

6. Let $A$ be a 6 by 6 nilpotent matrix over the complex numbers.
   a) (5 pts) Prove that $A^6 = 0$.
   b) (10 pts) Assume that $A^3 = 0$ and $A^2 \neq 0$. Find all possible Jordan canonical forms for such a matrix.
1. Let $G$ be a group and $N_1, \ldots, N_k$ subgroups of $G$.
(a) What conditions must be met in order for $G$ to be the internal direct product of $N_1, \ldots, N_k$?
(b) If $G$ is the internal direct product of $N_1, \ldots, N_k$, show $G \cong N_1 \times \cdots \times N_k$.

2. Let $(G, +)$ be a finite abelian group and $n$ a positive integer. Let $\phi : G \to G$ be the abelian group homomorphism defined by $\phi(g) = ng$, for all $g \in G$. Here, we are writing $ng$ for $g + \cdots + g$, $n$ times. Prove that $\ker(\phi) = G/\text{im}(\phi)$.

3. Let $R$ be a unique factorization domain with quotient field $K$. If $f(z) \in R[z]$ is a primitive polynomial, show that $f(z) \cdot K[z] \cap R[z] = f(z) \cdot R[z]$.

4. Let $\epsilon := e^{2\pi i}$. (a) Show that $\epsilon$ satisfies $x^2 - x + 1 = 0$.
(b) Verify that $[\mathbb{Q}(\epsilon) : \mathbb{Q}] = 2$.
(c) Show that $\phi : \mathbb{Q}(\epsilon) \to \mathbb{Q}(\epsilon)$ defined by $\phi(a + bc) = (a + b) - bc$ is an automorphism. (Note, $a, b \in \mathbb{Q}$.)
(d) Explain why $id$ and $\phi$ are the only automorphisms of $\mathbb{Q}(\epsilon)$.

5. Let $V$ be the vector space of complex polynomials having degree five or less and $D : V \to V$ the linear transformation given by differentiation. Find the Jordan canonical form for $D^2 := D \circ D$.

6. Let $V$ be an $n$-dimensional vector space over the field $K$ and $T : V \to V$ a linear transformation.
(a) Define the characteristic polynomial $f_T(x)$ of $T$. Why is your answer well-defined?
For parts (b) and (c) assume there exists $v \in V$ such that $\{v, T(v), \ldots, T^{n-1}(v)\}$ is a basis for $V$.
(b) Show that the matrix of $T$ with respect to the given basis equals the companion matrix of $f_T(x)$.
(c) Use the given basis to show that the conclusion of the Cayley-Hamilton theorem holds for $T$, i.e., $f_T(T) = 0$.
(e) How can the foregoing ideas be used to give a proof of the Cayley-Hamilton theorem in general?
1. For \( n \geq 1 \), let \( D_n \) denote the dihedral group (i.e., the group of symmetries of a regular \( n \)-sided polygon).

(a) Define \( D_n \) in terms of generators and relations.
(b) Show that \( D_4 \) is not isomorphic to \( Q_8 \), the quaternion group.
(c) Does \( D_9 \) have an element of order 4? Justify your answer.

2. Let \( G \) be a finite group and suppose there exists an element in \( G \) which has only two distinct conjugates. Show that \( G \) has a non-trivial normal subgroup.

3. Let \( F \) be a field and \( F[X] \) denote the ring of polynomials with coefficients in \( F \).

(a) State the division algorithm, as it applies to elements of \( F[X] \).
(b) Using the division algorithm, give a procedure for finding the greatest common divisor of any two non-zero elements of \( F[X] \).
(c) Let \( F := \mathbb{Z}_5 \), \( f(X) := X^5 + X^4 + X^3 + X^2 + X + 1 \) and \( g(X) := X^3 + 4 \). Find the GCD of \( f(X) \) and \( g(X) \).
(d) Find \( l(X), h(X) \in \mathbb{Z}_5[X] \) such that \( \text{GCD}(f, g) = l(X) \cdot f(X) + h(X) \cdot g(X) \).

4. Let \( V \) be a finite dimensional inner product space over \( \mathbb{C} \) and \( T : V \rightarrow V \) a normal linear transformation, i.e., \( TT^* = T^*T \), where \( T^* \) denotes the Hermitian adjoint of \( T \). Prove that if \( T^k = 0 \), some \( k \), then \( T = 0 \). (If you prefer, you may prove the corresponding statement for square matrices over \( \mathbb{C} \).)
5. Describe the 'canonical form' taken by nilpotent matrices, i.e., if $A$ is a square nilpotent matrix over the field $F$, describe its rational (or Jordan) canonical form. Find the nilpotent form for the matrix

$$
\begin{pmatrix}
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1/2 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$
1. Define unique factorization domain (UFD). Let $R$ be a UFD. Assume that $\alpha$ belongs to the quotient field of $R$ and satisfies a monic polynomial with coefficients in $R$. Show that $\alpha \in R$.

2. Fix an integer $n > 1$ and let $X \subseteq \mathbb{C}$ denote the roots of the polynomial $x^n - 1$.
   (a) Show that $X$ is a cyclic group of order $n$ under complex multiplication.
   (b) A generator of $X$ is called a primitive $n$th root of unity. How many primitive $n$th roots of unity are there?
   (c) Let $\epsilon$ be a primitive $n$th root of unity and $\sigma, \tau$ automorphisms of the field $\mathbb{Q}(\epsilon)$. Show that $\sigma \tau = \tau \sigma$.
   (d) Let $\epsilon$ be a primitive cube root of unity. Show by explicit calculation that there exists an automorphism $\sigma$ of $\mathbb{Q}(\epsilon)$ taking $\epsilon$ to $\epsilon^2$.

3. Let $G$ be a finite group and $p$ a prime number dividing $|G|$. Let $S$ denote the set of $p$-tuples $a$ of elements of $G$ such that the product of the coordinates of $a$ equals $e$. That is, $S := \{(g_1, \ldots, g_p) \mid g_i \in G \text{ and } g_1 g_2 \cdots g_p = e\}$.
   (a) Show that if $a \in S$, then any cyclic permutation of $a$ belongs to $S$. I.e., if $(g_1, \ldots, g_p)$ belongs to $S$, then $(g_{i}, g_{i+1}, \ldots, g_p, g_1, \ldots, g_{i-1})$ belongs to $S$ for all $1 \leq i \leq p$.
   (b) For $a, b \in S$, define $a \sim b$ if $b$ is a cyclic permutation of $a$. Show that $\sim$ is an equivalence relation on $S$.
   (c) Let $[a]$ denote the equivalence class of $a \in S$. Show that $|[a]| = 1$ or $|[a]| = p$.
   (d) Calculate $|S|$.
   (e) Use the foregoing to show that $G$ has an element of order $p$. 
4. Let $V$ be a finite dimensional vector space and $T : V \to V$ a linear transformation satisfying $\text{rank}(T) = \text{rank}(T^2)$. Show that $V = \ker(T) \oplus \text{range}(T)$.

5. State either the Jordan canonical form theorem or the rational canonical form theorem. Find the Jordan canonical form AND the rational canonical form for the matrix

$$
\begin{pmatrix}
0 & 3 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 10 & 0 & -3 \\
1 & 0 & 1 & 0
\end{pmatrix}
$$