Qualifying Exam in Analysis, January 2009

Instructions: Work ALL the following six problems. Write clearly on one side of your paper and write your name on every sheet that you use.

1) Let \( \{a_n\}_n \) be a sequence of points in a metric space \( X \) with metric \( \rho \). Suppose that the subsequences \( \{a_{2k}\}_k, \{a_{2k+1}\}_k, \{a_{3k}\}_k \) are convergent. Show that the whole sequence \( \{a_n\} \) is convergent.

2) For a real valued function \( f \) defined in an open neighborhood of \( x \in \mathbb{R} \), define

\[
D^+ f(x) = \limsup_{n \to \infty} (n(f(x + 1/n) - f(x)))
\]

(where the expression in the \( \limsup \) is clearly well-defined for \( n \in \mathbb{N} \) large). Show that if \( f : [a, b] \to \mathbb{R} \) is continuous and there exists \( \epsilon > 0 \) such that \( D^+ f(x) > \epsilon \) for all \( x \in (a, b) \), then \( f(x) \leq f(y) \) for all \( x, y \in [a, b] \), \( x \leq y \).

3) Let \( f : [a, b] \to \mathbb{R} \) be a bounded monotone function. Show that \( f \) is Riemann integrable on \([a, b]\). (Remark: You cannot invoke a result from measure theory that states that a bounded function is Riemann integrable iff the set of discontinuity has measure zero. Such a result is not needed for this exercise.)

4) Let \( f, g : [a, b] \to \mathbb{R} \) be continuous functions with \( |g(x)| > 0 \) for all \( x \in [a, b] \). Show that if \( \{f_n\}_n \) and \( \{g_n\}_n \) are sequences of functions (not necessarily continuous) that converge uniformly to \( f \) and \( g \), then \( \{f_n/g_n\}_n \) converges uniformly to \( f/g \).

5) Let \( U, V \) be open sets in \( \mathbb{R}^n \) and assume that \( V \) is connected. Suppose that \( f : U \to V \) is a \( C^1 \) function whose Jacobian determinant is never zero and such that \( f^{-1}(K) \) is compact in \( \mathbb{R}^n \) for all compact sets \( K \) contained in \( V \). Show that \( f(U) = V \). (Hint: Show that \( f(U) \cap V = f(U) \cap V \).

Use then the connectivity of \( V \).)

6) Prove the following Logarithmic Test for convergence of series. Suppose that \( \{a_k\} \) is a sequence of real numbers such that \( a_k \neq 0 \) for large \( k \) and

\[
p = \lim_{k \to \infty} \frac{\log(1/|a_k|)}{\log k}
\]

exists (possibly as an extended real number). Show that if \( p > 1 \) then the series \( \sum_{k=1}^{\infty} |a_k| \) is convergent and if \( p < 1 \) then \( \sum_{k=1}^{\infty} |a_k| \) is divergent.
INSTRUCTIONS: Work ALL the following six problems. Write clearly on one side of your paper and write your name on every sheet that you use.

1. Let \( f : [a, b] \to \mathbb{R} \) be a continuous function. Show that the functions

\[
M(x) = \sup_{a \leq \xi < x} f(\xi)
\]

and

\[
m(x) = \inf_{a \leq \xi < x} f(\xi)
\]

are also continuous.

2. Let \( f : \mathbb{R} \to \mathbb{R} \) be a function whose derivative \( f' \) exists on the interval \((a, b)\), and let

\[
g_n(x) = n(f(x + 1/n) - f(x)).
\]

Prove that the sequence of functions \( \{g_n\}_{n=1}^\infty \) converges uniformly to \( f' \) on every interval \([a_1, b_1] \subset (a, b)\) if and only if \( f' \) is continuous on \((a, b)\).

3. Show that the series

\[
\sum_{n=1}^\infty \frac{(-1)^n}{\sqrt{n^2 + x^4}}
\]

converges uniformly on all \( \mathbb{R} \) but is not absolutely convergent for any \( x \) in \( \mathbb{R} \).

4. Give a proof of the following result. If \( f \) is a real-valued continuous function on the finite rectangle \([a, b] \times [c, d]\), then the function

\[
F(y) = \int_a^b f(x, y) \, dx
\]

is well-defined (meaning the above integral is defined) and continuous on \([c, d]\).

5. Let \( \{K_j\}_{j=1}^\infty \) be a collection of nonempty compact sets in a metric space \( X \). Assume that \( K_{j+1} \subseteq K_j \) for all \( j \geq 1 \), and suppose that \( E \) is a nonempty closed set in \( X \) such that \( E \cap (\bigcap_{j=1}^\infty K_j) = \emptyset \). Show that there exists an integer \( N \geq 1 \) such that \( E \cap K_j = \emptyset \) for all \( j \geq N \).

6. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be given by

\[
f(x, y) = \begin{cases} 
\frac{x^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\
0 & \text{if } (x, y) = (0, 0).
\end{cases}
\]

Show that the first-order partial derivatives of \( f \), \( f_x \) and \( f_y \), exist at \((0, 0)\). Show, however, that \( f \) is not differentiable at \((0, 0)\).
Qualifying Exam in Analysis, May 2008

Instructions: Work ALL the following six problems. Write clearly on one side of your paper and write your name on every sheet that you use.

1) Suppose that $f$ is a real valued continuous function on $[0, \infty)$, differentiable on $(0, \infty)$, such that $f(0) = 0$, and $f'$ is increasing. Show that for all $0 < x < y$ \[
\frac{f(x)}{x} < \frac{f(y)}{y}.
\]

2) Show that for any two positive and bounded sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$, it holds that \[
\lim_{n \to \infty} \sup (a_n b_n) \leq (\lim_{n \to \infty} \sup a_n) (\lim_{n \to \infty} \sup b_n).
\]
Show an example in which the inequality is strict.

3) Let $X$ be a metric space with metric $\rho$, and let $K_1$ and $K_2$ be two compact sets in $X$ such that $K_1 \cap K_2 = \emptyset$. Show that there exist two open sets $U_1$ and $U_2$ such that $K_1 \subset U_1$, $K_2 \subset U_2$, and $U_1 \cap U_2 = \emptyset$.

4) Recall that $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at a point $a \in \mathbb{R}^2$ if there exists $B \in \mathbb{R}^2$ such that \[
\lim_{h \to (0,0)} \frac{|f(a + h) - f(a) - B \cdot h|}{\|h\|} = 0.
\]
a) Show that if $f$ is differentiable at $a$ then it is continuous at $a$.
b) Consider the function defined by $f(x, y) = xy^2(x^2 + y^4)^{-1}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Is $f$ differentiable at $(0, 0)$? Justify your answer.

5) Consider the system of equations
\[
x^2 + y^2 = z \\
x^2 + y^3 = w
\]
a) Show that there is an open set $W$ in $\mathbb{R}^2$ containing the point $(x_0, y_0) = (2, 2)$ and an open set $V$ in $\mathbb{R}^2$ containing the point $(x_0, y_0) = (1, 1)$ such that for each $(z, w)$ in $W$ there exists a unique solution $(x, y)$ in $V$ and that the solution depends continuously on the data $(z, w)$.
b) For $(z, w)$ in $W$, is the solution unique in all of $\mathbb{R}^2$? Justify your answer.

6) a) Prove the following result about uniform convergence and integration: If $\{f_n\}_{n=1}^{\infty}$ is a sequence of real-valued Riemann integrable functions on the interval $[a, b]$ converging uniformly in $[a, b]$ to a function $f$, then $f$ is Riemann integrable on $[a, b]$.
b) Show with an example that the result is not true if uniform convergence is replaced by pointwise convergence.
INSTRUCTIONS: Work ALL the following six problems. Write clearly on one side of your paper and write your name on every sheet that you use.

1. We say that a function \( f : [a, b] \rightarrow \mathbb{R} \) is of bounded variation, if the following quantity

\[
\text{Var}(f, [a, b]) := \sup_{a = x_0 < x_1 < \ldots < x_n = b} \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|
\]

is finite. Show that if \( f \) is differentiable and \(|f'|\) is Riemann integrable, then \( f \) is of bounded variation and

\[
\text{Var}(f, [a, b]) = \int_{a}^{b} |f'(x)| \, dx.
\]

2. Let \( \{a_n\} \) be a sequence, such that \( \sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty \). Show that \( \{a_n\} \) converges. Is the converse true or false? That is, given a convergent sequence \( \{a_n\} \), is it always true that \( \sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty \)? Prove or provide a counterexample.

3. Let \( f : [0, \infty) \rightarrow \mathbb{R}, f \geq 0 \) and \( \int_{0}^{\infty} f(x) \, dx < \infty \), that is, \( f \) is Riemann integrable in any finite interval \([0, R]\) and \( \lim_{R \to \infty} \int_{0}^{R} f(x) \, dx \) exists. Show that

\[
\lim_{R \to \infty} \frac{\int_{0}^{R} xf(x) \, dx}{R} = 0.
\]

4. Let \( \mathcal{U} \) be an open covering of a compact set \( D \) in a metric space \( X \). Show that there is a number \( \epsilon > 0 \) such that if \( p \) and \( q \) are in \( D \) and \( d(p, q) < \epsilon \), then \( p, q \in V \) for some \( V \in \mathcal{U} \).

5. Give an example of a sequence of continuous functions \( f_n : [0, 1] \rightarrow \mathbb{R} \) converging pointwise on \([0, 1]\) to a continuous functions \( f \) and such that \( \int_{0}^{1} f_n(x) \, dx = 1 \) for all \( n \) and \( \int_{0}^{1} f(x) \, dx = 0 \).

6. Show that the series \( \sum_{n=1}^{\infty} \left( \frac{x}{n} - \ln(1 + \frac{x}{n}) \right) \) converges uniformly on \((-1, A)\) for \( A > -1 \) and that the sum of the series has derivatives of all orders on \((-1, +\infty)\).
INSTRUCTIONS: Work ALL the following six problems. Write clearly on one side of your paper and write your name on every sheet that you use.

1.- For $a \in \mathbb{R}$, consider the set of points in the plane defined by $C_a = \{(x,y) : x^3 + y^3 = 3axy\}$. For which values of $a$ is $C_a$ locally the graph of a $C^1$ function of one variable? Justify your answer.

2.- Let $X$ be a metric space and let $f : S \subset X \to \mathbb{R}$ be a continuous function.
   (a) If $S$ is a compact set, prove that $f$ is uniformly continuous on $S$.
   (b) Show that $f$ may fail to be uniformly continuous if $S$ is only assumed to be closed.

3.- Prove the following version of the First Mean Value Theorem for Integrals:
   If $f$ is continuous on $[a,b]$ and $g$ is a positive Riemann integrable on $[a,b]$, then there exists a $\xi \in [a,b]$ such that
   $$\int_{a}^{b} f(x)g(x)\,dx = f(\xi) \int_{a}^{b} g(x)\,dx.$$

4.- Prove the integral test for convergence: Let $f$ be a Riemann integrable positive monotone decreasing function on $[1,\infty)$. Define $F(x) = \int_{1}^{x} f(t)\,dt$ for $x \in [1,\infty)$ and $a_n = f(n), n \geq 1$. Then $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\lim_{x \to \infty} F(x)$ exists.

5.- (a) Show that for $0 < a < \pi$, $\lim_{n \to \infty} \int_{0}^{\pi} \frac{\sin nx}{nx}\,dx = 0$.
   (b) What is $\lim_{n \to \infty} \int_{0}^{\pi} \frac{\sin nx}{nx}\,dx$? Justify your answer.

6.- Let $f$ be a bounded function defined on $(-\delta_0, \delta_0)$ for some $\delta_0 > 0$. Define the 'saltus' of $f$ at 0, by
   $$S(f,0) = \inf_{\delta \leq \delta_0} \left\{ \sup_{x \in (-\delta,\delta)} f(x) - \inf_{x \in (-\delta,\delta)} f(x) \right\}.$$  
   Show that if for some $L > 0$, $|f(x) - f(y)| < L$ for all $x, y$ in $(-\delta_0, \delta_0)$, then $S(f, 0) \leq L$. 


INSTRUCTIONS: Work ALL the following six problems. Justify all your steps. Write clearly on one side of your paper and write your name on every sheet that you use.

1.- Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function. Assume that \( f(0) = 0, f'(0) < 0 \) and \( \lim_{x \to \infty} f(x) = 1 \). Show that there is an \( x_0, 0 < x_0 < \infty \) such that \( f(x_0) = 0 \).

2.- Let \( f : [0, 2] \to \mathbb{R} \) be a Riemann integrable function. Show that
\[
\lim_{n \to \infty} \int_0^1 f \left( x + \frac{1}{n} \right) dx = \int_0^1 f(x)dx.
\]
Notice that \( f \) is not assumed to be continuous.

3.- Let \( f \) be differentiable on the interval \((a, b)\). Show that \( f' \) is bounded if and only if \( f \) satisfies the following "Lipschitz condition": there exists a constant \( M > 0 \), so that
\[
|f(x) - f(y)| \leq M|x - y|,
\]
for all \( x, y \in (a, b) \).

4.- Let \( F \) be a \( C^1 \) function, \( F = (u, v) : U \subset \mathbb{R}^2 \to \mathbb{R}^2 \), where \( U \) is an open subset of \( \mathbb{R}^2 \). Assume that \( F(x, y) = (u(x, y), v(x, y)) \) satisfies the Cauchy-Riemann equations, i.e.,
\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.
\]
Show that if at least one of the first partial derivatives of either \( u \) or \( v \) is different from zero at a point \((x_0, y_0) \in U\), then there is a neighborhood \( W \) of \((x_0, y_0) \), \( W \subset U \), such that \( F|_W \) has a \( C^1 \) inverse defined on \( F(W) \).

5.- Let \( C[0, 1] \) be the space of real valued continuous function on \([0, 1]\). Fix \( \psi \in C[0, 1] \) and define
\[
\rho_{\psi}(f, g) := \int_0^1 \psi(x)|f(x) - g(x)|\,dx, \ f, g \in C[0, 1].
\]
(i) Show that if \( \psi(x) > 0 \) for all \( x \in [0, 1] \), then \( \rho_{\psi} \) is a metric on \( C[0, 1] \).
(ii) Show that if \( \psi(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2} \\ x - \frac{1}{2}, & \frac{1}{2} < x \leq 1 \end{cases} \) then \( \rho_{\psi} \) is not a metric on \( C[0, 1] \).

6.- Let \( F : [0, 1] \times [0, 1] \to \mathbb{R} \) be a continuous function. Define
\[
F^*(x) = \sup\{F(x, t) : 0 \leq t \leq 1\}.
\]
Show that \( F^* \) is a continuous function on \([0, 1]\).
INSTRUCTIONS: Work ALL the following six problems. Write clearly on one side of your paper and write your name on every sheet that you use.

1.- Prove the following version of Taylor’s theorem. Let \( f \) be a function on \( \mathbb{R} \) with second derivative at every point. Show that for each \( x \) in \( \mathbb{R} \) there exists a \( c \) between 0 and \( x \) such that

\[
f(x) = f(0) + f'(0)x + \frac{f''(c)}{2}x^2.
\]

2.- Prove the following part of the Fundamental Theorem of Calculus. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a continuous functions and define \( F(x) = \int_a^x f(t)dt \). Then, \( F \) is differentiable in \( (a, b) \) and

\[
\frac{d}{dx} \int_a^x f(t)dt = f(x), \text{ for each } x \in (a, b).
\]

3.- Let \( X \) be a metric space. A family \( \{E_a\}_{a \in A} \) of subsets of \( X \) is said to have the finite intersection property if \( \cap_{a \in A} E_a \neq \emptyset \) for all finite sets \( B \subset A \). Show that \( X \) is compact if and only if every family \( \{E_a\}_{a \in A} \) of closed sets with the finite intersection property satisfies \( \cap_{a \in A} E_a \neq \emptyset \).

4.- Let \( f \) be a bounded function on an interval \([a, b] \). Suppose that there exists a sequence of points \( \{x_n\} \) contained in \([a, b] \) converging to a point \( x_0 \) and such that \( f \) is continuous on \([a, b] \setminus \{x_n\} \). Prove that \( f \) is Riemann integrable on \([a, b] \).

Remark: You cannot invoke a more advanced result in integration theory which states that a bounded function on an interval \([a, b] \) is Riemann integrable if and only if its set of discontinuities has Lebesgue measure zero.

5.- Consider the set of points \( S \) given by \( y^2z = x^2 \) (called Whitney’s umbrella). For which points \( P \) on \( S \) is there a neighborhood of \( P \), \( V_P \), and an open set \( D \) in \( \mathbb{R}^2 \) such that \( V_P \cap S \) is diffeomorphic to \( D \)? Justify your arguments.

6.- Give an example of a sequence of differentiable functions \( f_n : \mathbb{R} \rightarrow \mathbb{R} \) such that \( f_n \) converges uniformly in all \( \mathbb{R} \) to a differentiable function \( f \) and such that \( f'(x) = \lim_{n \to \infty} f'_n(x) \) if \( x \neq 0 \) but \( f'(0) \neq \lim_{n \to \infty} f'_n(0) \).
Qualifying Exam in Analysis, August 8, 2006

Instructions: Solve ALL of the following six problems. Write down clearly your name (at the upper right corner) and your solution on EXACTLY ONE SIDE of each sheet that you use. Start each problem on a new page, and order all pages according to the problem number. Clip, but do NOT staple, your solution pages together.

1. If a real-valued function \( f \) is differentiable on a neighborhood of a point \( a \in \mathbb{R} \) and \( \lim_{x \to a^+} f'(x) \) exists, show that \( \lim_{x \to a^+} f'(x) = f'(a) \).

2. Show that the series \( \sum_{k=1}^{\infty} \frac{x^k}{k} \) converges uniformly on every interval of the form \([0, b]\) with \( b \in (0, 1)\), but not on \([0, 1]\).

3. Show that if \( f : [a, b] \to \mathbb{R} \) is continuous and increasing, then \( \sup f(E) = f(\sup E) \) for every nonempty set \( E \subset [a, b] \).

4. Show that if \( Y \) is a dense subset of a metric space \((X, d)\) such that every Cauchy sequence \((y_n)_{n \in \mathbb{N}}\) in \( Y \) converges in \( X \), then \( X \) is complete.

5. (i) Let \( f : [0, 1] \to \mathbb{R} \) be a bounded function. Prove the statement that if there are Riemann integrable functions \( f_n : [0, 1] \to \mathbb{R} \) converging to \( f \) uniformly on every interval \([a, b] \subset (0, 1)\), then \( f \) is Riemann integrable on \([0, 1]\).

   (ii) The above statement in (i) is no longer valid if uniform convergence is replaced by pointwise convergence. Illustrate this by showing a concrete sequence of Riemann integrable functions \( f_n \) converging pointwise to a bounded non-Riemann integrable function \( f \) on \([0, 1]\).

6. Let \( X \) be the subset of \( \mathbb{R}^4 \) defined by \( x_1 x_2^2 x_3 - x_1 x_3^3 - 2x_4^2 = -2, x_1 x_3^2 - x_3 + x_1 x_2^3 x_4 = 1, \) and \( x_i > 0 \) for all \( i \). Show that the map \( p : X \to \mathbb{R}^2 \) defined by \( p(x_1, x_2, x_3, x_4) = (x_1, x_3) \) is an open map, i.e. the image set \( p(V) \) is open for every open subset \( V \) of \( X \), or more explicitly, \( p(U \cap X) \) is open for every open subset \( U \) of \( \mathbb{R}^4 \).
Qualifying Exam in Analysis, May 23, 2006

Instructions: Solve ALL of the following six problems. Write down your name and solution clearly on EXACTLY ONE SIDE of each sheet that you use. Start each problem on a new page, and order the pages according to the problem number. Clip, but do NOT staple, your solution pages together.

1. Let \( f : (a, b) \to \mathbb{R} \) be a twice differentiable function (i.e. assume that \( f'' : (a, b) \to \mathbb{R} \) is well defined). Show that if \( f(x_2) < \min \{ f(x_1), f(x_3) \} \) for some \( x_i \in (a, b) \) with \( a < x_1 < x_2 < x_3 < b \), then there is a point \( c \in (a, b) \) such that \( f''(c) > 0 \).

2. (i) Assume that \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\), differentiable on \((a, b)\), and that \( M = \sup \{|f'(x)| : x \in (a, b)\} \) is finite. Prove that

\[
\int_a^b |f(a) - f(x)| \, dx \leq \frac{M(b-a)^2}{2}.
\]

(ii) Assume that the function \( f : \mathbb{R} \to \mathbb{R} \) is differentiable on \( \mathbb{R} \) and set \( M = \sup \{|f'(x)| : x \in (0, 1)\} \). Prove that, for each positive integer \( n \),

\[
\left| \frac{1}{n} \sum_{j=0}^{n-1} \frac{f(j/n)}{n} - \int_0^1 f(x) \, dx \right| \leq \frac{M}{2n}.
\]

3. A fundamental result about Riemann integration states that \( g \circ f \) is Riemann integrable for any continuous function \( g : [c, d] \to \mathbb{R} \) and any Riemann integrable function \( f : [a, b] \to [c, d] \).

Prove directly a special case of this, namely, \( f^2 \) is Riemann integrable for any nonnegative Riemann integrable function \( f \geq 0 \) on \([a, b]\), where \( f^2(x) := (f(x))^2 \). Hint: Compare/relate \( U(f^2, P) - \int L(f^2, P) \) to \( U(f, P) - \int L(f, P) \) for any partition \( P \) of \([a, b]\). Note that \( s^2 - t^2 = (s-t)(s+t) \).

4. Let \((X, d)\) be a metric space and \( f \) be a real-valued function defined on a neighborhood \( U \) of a point \( a \in X \). Recall that a real number \( L \) is called the limit of \( f(x) \) at \( a \) and denoted as \( \lim_{x \to a} f(x) \), if for any \( \epsilon > 0 \), there is \( \delta > 0 \) such that \( 0 < d(x, a) < \delta \) implies \( |f(x) - L| < \epsilon \). (Such a limit \( L \) if exists is well known to be unique.) Show that \( \lim_{x \to a} f(x) \in \mathbb{R} \) exists if \( f(x_n) \) is a Cauchy sequence for any sequence \( x_n \in U \setminus \{a\} \) with \( \lim_{n \to \infty} x_n = a \).

5. (i) Find a sufficient condition for a point \((x_0, y_0, z_0)\) in the intersection \( C \) of surfaces \( x^2 + y^2 + z^2 = 1 \) and \( x^2 + y^3 + z^3 = 0 \) to have a neighborhood \( U \) with \( U \cap C \) parametrized by the \( x \)-coordinate as a curve, i.e. \( U \cap C = \{(x, f(x)) : x \in (a, b)\} \) for some interval \((a, b)\) and some function \( f : (a, b) \to \mathbb{R}^2 \).

(ii) Then in terms of \( x, y, z \), compute the implicit derivative \( dz/dx \) at points \((x, y, z)\) sufficiently close to \((x_0, y_0, z_0)\) on such a curve \( U \cap C \).

6. Show that if \( \sum_{n=1}^{\infty} f_n \) converges pointwise to a continuous function \( f \) on \([0, 1]\) for some continuous functions \( f_n \geq 0 \) on \([0, 1]\), then \( \sum_{n=1}^{\infty} f_n \) converges uniformly to \( f \) on \([0, 1]\).
Qualifying Exam in Analysis, January 10, 2006

Instructions: Solve ALL of the following six problems. Write down your name and solution clearly on EXACTLY ONE SIDE of each sheet that you use. Start each problem on a new page, and order the pages according to the problem number. Clip, but do NOT staple, your solution pages together.

1. Does the sequence \( (f_n)_{n=1}^{\infty} \) of functions defined by \( f_n(x) = n^2 x^n (1 - x) \) converge uniformly to \( f = 0 \) on \([0,1]\)? Why or why not?

2. Show that common solutions of the equations \( t^3 + x^3 + y^3 + z^3 = 1 \), \( t^2 + x^2 + y^2 + z^2 = 3 \), and \( 2 + x + y + z = 2 \) near (i.e. in a neighborhood of) the point \((t_0, x_0, y_0, z_0) = (1, 1, -1, 0)\) form a curve parametrized by \((t, x, y, z) = (t, \gamma(t))\) with \(t \in I\), where \( I \) is an open interval containing 1 in \(\mathbb{R}\) and \(\gamma : I \to \mathbb{R}^3\) is a differentiable function with \(\gamma(1) = (1, -1, 0)\).

3. Let \((X, d)\) be a complete metric space. Complete the following steps (which form a proof of the well-known Banach Contraction Principle).
   
   (i) Show that if a sequence \((x_n)_{n=0}^{\infty}\) in \(X\) is contracting in the sense that there is a nonnegative constant \(c < 1\) such that \(d(x_{n+1}, x_n) \leq cd(x_n, x_{n-1})\) for all \(n \in \mathbb{N}\), then it converges to a point \(x\) in \(X\).

   (ii) Show that if \(f : (X, d) \to (X, d)\) is a contraction on \(X\), i.e. there is a nonnegative constant \(c < 1\) such that \(d(f(x), f(y)) \leq cd(x, y)\) for all \(x, y \in X\), then for any \(x_0 \in X\), the sequence \((x_n)_{n=0}^{\infty}\) inductively defined by \(x_{n+1} := f(x_n)\) converges to a point \(x \in X\), and this limit point \(x\) is the unique solution (independent of \(x_0\)) of the equation \(x = f(x)\).

4. Given a continuous function \(f : \mathbb{R} \to \mathbb{R}\), we define

\[ f_n(x) := n \int_{x - \frac{1}{n}}^{x + \frac{1}{n}} f(y) \, dy. \]

Answer each of the following questions by providing either a proof or an explicit counterexample. (You may assume that \(f\) is continuously differentiable if you need this stronger property.)

(i) Does the sequence \((f_n)_{n=1}^{\infty}\) converge uniformly to \(f\) on every compact subset \(K\) of \(\mathbb{R}\)?

(ii) Does the sequence \((f_n)_{n=1}^{\infty}\) converge uniformly to \(f\) on all of \(\mathbb{R}\)?

5. Let \(f\) be a real-valued differentiable function on an open set containing a finite interval \([a, b]\) where \(-\infty < a < b < \infty\). (Note that \(f'\) is not assumed to be continuous.)

(i) Show that if \(f'(a) < 0 < f'(b)\), then there is \(c \in (a, b)\) such that \(f'(c) = 0\). (Hint: Relate to some idea from elementary calculus.)

(ii) Show that more generally, we have the Darboux's Theorem: For any \(m \in \mathbb{R}\) with \(f'(a) < m < f'(b)\), then there is \(c \in (a, b)\) such that \(f'(c) = m\). (Hint: Suitably modify \(f\) by a linear function.)

6. Show that \(\sum_{n=1}^{\infty} a_n b_n\) converges, if \(\sum_{n=1}^{\infty} a_n\) converges and \(\sum_{n=1}^{\infty} (b_{n+1} - b_n)\) absolutely converges, where \(a_n, b_n \in \mathbb{R}\). (Hint: With \(b_0 = 0\), let \(c_n = b_n - b_{n-1}\). Given \(M < N\), first show by one intermediate step that

\[ \sum_{n=M}^{N} a_n b_n = \sum_{n=M}^{N} \sum_{k=1}^{N} \tau_{n,k} a_n c_k, \]

where \(\tau_{n,k} = 1\) if \(k \leq n\) and \(\tau_{n,k} = 0\) if \(k > n\), and then interchange \(\sum_{n=M}^{N}\) and \(\sum_{k=1}^{N}\).
QUALIFYING EXAM IN ANALYSIS
9 August 2005

Work all six problems. Use 8 1/2 × 11 paper, one side per sheet. Number the pages. Sign or initial each page.

1. For \( n = 1, 2, \ldots \) define \( f_n \) on \( \mathbb{R} \) by \( f_n(x) = (nx^2 + 1)^{-1} \), and let \( f(x) = \lim_n f_n(x) \). Find \( f(x) \). Is the convergence uniform on \([1, +\infty)\)? Why or why not? Is the convergence uniform on \([-1, 1]\)? Why or why not?

2. Let \( u, v : \mathbb{R}^2 \to \mathbb{R} \) be continuously differentiable functions. Let \( C = \{(x, y) : v(x, y) = 0\} \). Use either the inverse function theorem or the implicit function theorem to show that if the restriction of \( u \) to \( C \) has a local maximum or minimum at \((x_0, y_0)\) in \( C \), then the gradient vectors \( \nabla u(x_0, y_0) \) and \( \nabla v(x_0, y_0) \) must be linearly dependent (that is, one of these vectors must be a scalar multiple of the other).

3. Let \((X, d)\) be a complete metric space, and let \( \{x_1, x_2, \ldots\} \) be a sequence in \( X \) with \( d(x_k, x_{k+1}) \leq 1/k^2 \). Show that \( \{x_k\} \) converges. Give an example with \( d(x_k, x_{k+1}) \leq 1/k \) such that \( \{x_k\} \) does not converge.

4. Evaluate

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{s^k}{(n-k)!}
\]

in closed form for \(|s| < 1\) by interchanging the order of summation. Briefly explain why it is legitimate to do this.

5. Show that if the integral of a nonnegative continuous real function over \([a, b]\) is zero, then the function must vanish identically on \([a, b]\).

6. Let \( \{g_1, g_2, \ldots\} \) be a pointwise bounded sequence of continuous real functions on a metric space \((X, d)\). Define \( f : X \to \mathbb{R} \) by \( f(x) = \sup_n g_n(x) \).

(a) Show that \( \{x : f(x) > r\} \) is open for each \( r \) in \( \mathbb{R} \).

(b) Show that if a sequence \( \{x_1, x_2, \ldots\} \) converges to \( x \) in \( X \), then

\[
\liminf_{k} f(x_k) \geq f(x).
\]

(c) Use (a) or (b) to show if \( X \) is compact, then \( f \) has a minimum value on \( X \).
Qualifying Exam in Analysis
24 May 2005

Work all six problems below. (Over ⇒ for problem #6.)

1. (a) Show that a uniformly continuous map between metric spaces takes Cauchy sequences to Cauchy sequences.

(b) Show that if \( f : [0, 1] \to \mathbb{R} \) is uniformly continuous, then the limit

\[
\lim_{x \to 0^-} f(x)
\]

exists.

2. The lim sup of a bounded real sequence \( \{x_n\} \) can be defined as

\[
\lim_n (\sup\{x_k : k \geq n\}) .
\]

It can also be defined as

\[
\max\{x : x_{n_k} \to x \text{ for some subsequence } \{x_{n_k}\}\}.
\]

Taking for granted that the set of subsequential limits of \( \{x_n\} \) is closed, show that these two definitions are equivalent.

3. Suppose the function \( f = f(t) \) is given by a power series

\[
f(t) = c_1 t + c_2 t^2 + \ldots
\]

that converges for all \( t \). Let \( a_1, a_2, \ldots \) be such that \( \sum_k |a_k| < \infty \). Show that the series \( \sum_k f(a_k t) \) converges for all \( t \), and that the resulting function is given by an everywhere convergent power series in \( t \). Write the coefficients of the power series in terms of the \( c \)'s and the \( a \)'s. (Hint: If \( |a_k| \leq M \), then \( |a_k^n| \leq |a_k| M^{n-1} \) for \( n = 1, 2, \ldots \).)

4. Let \( U \) be an open subset of \( \mathbb{R}^d \). Show that \( U \) is the union of countably many compact sets, and then show that \( U \) is the union of countably many sets of the form \( (a_1, b_2) \times (a_2, b_2) \times \ldots \times (a_d, b_d) \).

5. Let \( f, g : \mathbb{R} \to \mathbb{R} \) be continuously differentiable. Suppose that \( f(0) = g(0) = 0 \) and that \( f'(0) \neq 0 \). Show that there exist an open subset \( V \) of \( \mathbb{R}^2 \) containing \( (0,0) \) and a continuously differentiable real function \( \psi \) on \( V \) such that \( \psi(u, v) = 0 \) if and only if \( (u, v) = (f(x), g(x)) \) for some \( x \) near 0. (Hint: Consider \( H : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( H(x, y) = (f(x), g(x) + y) \).)
6. Let \( G : [0, 1] \times [0, 1] \to \mathbb{R} \) be continuous. Suppose also that the partial derivative \( G_1 \) of \( G \) with respect to the first variable exists and is continuous everywhere on \( [0, 1] \times [0, 1] \). Define \( f : [0, 1] \to \mathbb{R} \) by
\[
f(x) = \int_0^1 G(x, y) \, dy.
\]
Show that \( f \) is differentiable on \( [0, 1] \) with \( f' \) given by
\[
f'(x) = \int_0^1 G_1(x, y) \, dy.
\]
(Hint: Write \( G(x + h, y) - G(x, y) = G_1(\phi_h(x, y), y)h \), where \( |\phi_h(x, y) - x| < |h| \).)
Qualifying Examination in Analysis
Wednesday 12 January 2005

Work all six problems below. Please write on one side of each sheet of paper.

1. Show that if \( \{a_0, a_1, a_2, \ldots\} \) is a real sequence with limit 0, then
   \[
   \lim_{t \to +\infty} e^{-t} \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n = 0 .
   \]

2. Let \( f \) be a Riemann integrable function on \([0,1]\). For real numbers \( a < b \), let \( I_{[a,b]} \) be the indicator function of the interval \([a,b]\), i.e., 1 on \([a,b]\), 0 off \([a,b]\).

   (a) Show that for every \( \epsilon > 0 \), one can find a partition \( P = \{x_0, x_1, \ldots, x_n\} \) of \([0,1]\) and numbers \( y_1, \ldots, y_n \) such that
   \[
   \left| \int_0^1 f - \sum_{i=1}^{n} y_i I_{[x_{i-1}, x_i]} \right| < \epsilon .
   \]

   (b) Show that
   \[
   \lim_{r \to \infty} \int_0^1 f(x) \sin(rx) \, dx = 0 .
   \]

3. Let \( f : \mathbb{R}^2 \to \mathbb{R}^3 \) be defined by \( f(x,y) = (x + y^2, xy^2, x^2y) \).

   (a) Show that \( f \) is one-to-one in a neighborhood of the point \((1,1)\).

   (b) Show that there is a continuously differentiable real function \( \Phi = \Phi(u,v,w) \) defined in an \( \mathbb{R}^3 \) neighborhood of \((2,1,1)\) with \( \Phi(2,1,1) = 0 \) such that \( f \) parametrizes the level surface \( \Phi^{-1}(0) \) near \((2,1,1)\). (Hint: Consider \( F \) defined by \( F(x,y,z) = f(x,y) + xz^2 \) for an appropriate \( c \) in \( \mathbb{R}^3 \).)

4. Let \( \{a_1, a_2, \ldots\} \) be a real sequence satisfying \( a_j + a_k \geq a_{j+k} \) for all \( j, k \geq 1 \). Show that
   \[
   \frac{a_k}{k} \geq \limsup_{j \to \infty} \frac{a_j}{j}
   \]
   and deduce from this that \( \lim_j a_j/j \) exists. (Hints: \( a_j + na_k \geq a_{j+nk} \), \( N + k = (kN + 1) \cup \ldots \cup (kN + k - 1) \cup (kN + k) \).)

5. Show that if \( f : [0, +\infty) \to \mathbb{R} \) is continuous on \([0, +\infty)\) and differentiable on \((0, +\infty)\), with \( f(0) = 0 \) and \( f' \) strictly increasing on \((0, +\infty)\), then the function \( x \mapsto f(x)/x \) is strictly increasing on \((0, +\infty)\).

6. (a) Let \( E \) and \( F \) be respectively a nonempty closed subset and a nonempty compact subset of a metric space \((X, d)\) with \( E \cap F = \emptyset \). Show that
   \[
   \inf\{d(e, f) : e \in E, f \in F\} > 0 .
   \]

   (b) Give an example to show that the conclusion of (a) may fail if \( F \) is closed but not compact.
Qualifying Examination in Analysis
10 August 2004

Work all SIX problems below. Show your work.

1. Show that if the first-order partial derivatives of a real function $f$ on $\mathbb{R}^2$ exist everywhere on $\mathbb{R}^2$ and are bounded, then $f$ is continuous on $\mathbb{R}^2$.

2. Let $f : U \to V$ be a continuously differentiable map between open subsets of $\mathbb{R}^n$. Suppose that the Jacobian determinant of $f$ vanishes nowhere on $U$, that $f^{-1}(K)$ is compact for all compact subsets $K$ of $V$, and that $V$ is connected. Show that $f(U) = V$.

3. Let $\{f_n\}$ be a sequence of differentiable real functions on $\mathbb{R}$ converging pointwise to a function $g$ on $\mathbb{R}$. Show that if the sequence $\{f'_n\}$ of derivatives is uniformly bounded, then $g$ is uniformly continuous.

4. Suppose $f : [0, \infty) \to [0, \infty)$ satisfies $|f(s) - f(t)| \leq K|s - t|$ for some positive constant $K$ and all $s, t \geq 0$, and that

$$\int_0^\infty f(x) \, dx < \infty.$$ 

Show that $\lim_{x \to \infty} f(x) = 0$.

5. Prove Dini's theorem, namely, if $\{f_n\}$ is a sequence of continuous real functions on a compact metric space $X$ converging pointwise to 0 with $f_1 \geq f_2 \geq f_3 \geq \ldots$, then $f_n \to 0$ uniformly.

6. Show that if $\lim_n a_n = a$ and $\lim_n b_n = b$, and

$$c_n = \frac{1}{n}(a_1b_n + a_2b_{n-1} + \ldots + a_nb_1),$$

then $\lim_n c_n = ab$ where $a_n, b_n, a$, and $b$ are real numbers.
Qualifying Examination in Analysis
24 May 2004

Work ALL SIX problems below.

1. Let \((X, d)\) be a metric space and let \(E\) be a nonempty closed subset of \(X\). Show that the function \(f : X \rightarrow [0, +\infty)\) defined by \(f(x) = \inf\{d(x, y) : y \in E\}\) is continuous, and that \(f(x) = 0\) if and only if \(x \in E\).

2. Let \(\{f_n\}\) be a uniformly convergent sequence of uniformly continuous real functions on \(\mathbb{R}\). Show that the sequence \(\{f_n\}\) is equicontinuous in the sense that for every \(\epsilon > 0\) there exists \(\delta > 0\) such that \(|f_n(x) - f_n(y)| < \epsilon\) for all \(n\) whenever \(|x - y| < \delta\).

3. Let \(\{a_n\}_{n=0}^{\infty}\) be a convergent complex sequence with limit \(A\). Show that the power series \(\sum_{n=0}^{\infty} a_n x^n\) converges for \(|x| < 1\) and that

\[
\lim_{x \to 1^-} (1 - x) \sum_{n=0}^{\infty} a_n x^n = A,
\]

where the limit is taken over \(x\) in the real interval \([0, 1)\).

4. Show that there are continuously differentiable real functions \(u = u(x, y)\) and \(v = v(x, y)\), defined in a neighborhood \(W\) of \((1, 1)\) in \(\mathbb{R}^2\) satisfying

\[
u(x, y)^3 + x u(x, y) - 2y v(x, y) = 0 = v(x, y)^3 + y^2 v(x, y)^2 - 2x u(x, y)
\]

for all \((x, y)\) in \(W\) and \(u(1, 1) = v(1, 1) = 1\).

5. Let \(\phi\) and \(f\) be continuous real functions on \([0, +\infty)\). Suppose that \(\phi\) is non-negative and satisfies \(\int_0^{\infty} \phi(x) \, dx = 1\). Suppose that \(f\) is bounded. Show that

\[
\lim_{n \to \infty} n \int_0^{\infty} f(y) \phi(ny) \, dy = f(0).
\]

6. Let \(h : [0, 1] \times [0, 1] \to \mathbb{R}\) be continuous. Define \(\phi : [0, 1] \to \mathbb{R}\) by \(\phi(x) = \max\{h(x, y) : y \in [0, 1]\}\). Show that \(\phi\) is continuous.
Qualifying Examination in Analysis
20 January 2004

Work all six problems below.

1. (a) Show that there exist continuously differentiable real functions \( f \) and \( g \) defined in a neighborhood \( U \) of \((2,3)\) in \( \mathbb{R}^2 \) such that \( f(2,3) = 1 = g(2,3) \) and
   \[
   f(x, y)^3 + g(x, y)^2 = x \\
   f(x, y)^2 + 2g(x, y) = y
   \]
   for all \((x, y)\) in \( U \).

   (b) Evaluate the partial derivatives \( f_x, f_y, g_x, g_y \) at \((2,3)\).

2. Show that
   \[
   \int_0^1 f(x) \, dx = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n f(k/n)
   \]
   for all continuous real functions on \([0,1]\).

3. Show that if \( f \) is a twice-differentiable real function on \([0,1]\) such that
   \[
   \min_{0 \leq x \leq 1} f(x) < f(0) < \max_{0 \leq x \leq 1} f(x) \quad \text{and} \quad \min_{0 \leq x \leq 1} f(x) < f(1) < \max_{0 \leq x \leq 1} f(x),
   \]
   then \( f''(c) = 0 \) for some \( c \) in \((0,1)\).

4. Let \((X, d)\) be a compact metric space. Show that
   \[
   \liminf_n \liminf_m d(x_n, x_m) = 0
   \]
   for every sequence \( \{x_n\} \) in \( X \).

5. It is well known that the space \( C[a, b] \) of continuous real functions on \([a, b]\) is complete in the metric \( d \) defined by
   \[
   d(f, g) = \max_{a \leq x \leq b} |f(x) - g(x)|.
   \]
   Show that the space \( C^1[a, b] \) of continuously differentiable real functions on \([a, b]\) is complete in the metric \( \rho \) defined by \( \rho(f, g) = |f(a) - g(a)| + d(f', g') \).

6. Let \( f = f(x, y) \) be a continuous real function on \( \mathbb{R}^2 \) whose partial derivative \( f_x \) with respect to its first argument exists and is continuous on \( \mathbb{R}^2 \).

   (a) Show that
   \[
   \lim_{h \to 0} \max_{a \leq x \leq b} \left( \frac{f(x + h, y) - f(x, y)}{h} - f_x(x, y) \right) = 0
   \]
   for all \( a, b, x \).

   (b) Show that
   \[
   \frac{d}{dx} \int_a^b f(x, y) \, dy = \int_a^b f_x(x, y) \, dy.
   \]
Instructions: Work All SIX problems. Justify your work.

1. For real $\alpha$, define $f_\alpha : \mathbb{R}^2 \to \mathbb{R}$ by $f_\alpha(x, y) = xy(x^2 + y^2)^\alpha$ for $(x, y) \neq (0, 0)$, and $f_\alpha(0, 0) = 0$.
   (a) For which values of $\alpha$ is $f$ continuous at $(0,0)$?
   (b) For which values of $\alpha$ is $f$ differentiable at $(0,0)$?

2. Prove the well known fact that every compact subset of a metric space is closed.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. Show that if $f'$ and $f''$ are uniformly continuous on $\mathbb{R}$, then for every $\epsilon > 0$ there exists $\delta > 0$ such that
   $$|f''(x) - \frac{f(x + h) + f(x - h) - 2f(x)}{h^2}| < \epsilon$$
   for all $x$ whenever $0 < |h| < \delta$.

4. Show that the series
   $$\sum_{n=1}^{\infty} \left( \frac{x}{n} - \log(1 + \frac{x}{n}) \right)$$
   converges uniformly on $(-1, A)$ for all $A > -1$ and that the sum of the series has derivatives of all orders on $(-1, +\infty)$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and 1 - periodic. Assume that $\int_{0}^{1} f(x) \, dx = 0$. Show that
   $$\int_{1}^{\infty} \frac{f(x)}{x} \, dx$$
   is convergent.
   (Hint: Consider an antiderivative of $f(x)$ and integration by parts.)

6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (x^2 - 2x - y^2, xy - y)$. Show that there is exactly one point $(x^*, y^*)$ with the following property: there exist sequences $\{(x_n, y_n)\}$ and $\{(x'_n, y'_n)\}$ converging to $(x^*, y^*)$ with $(x_n, y_n) \neq (x'_n, y'_n)$ and $f(x_n, y_n) = f(x'_n, y'_n)$ for every $n$. 


Instructions: Work All SIX problems. Justify your work.

1. Let \( f : \mathbb{R} \to \mathbb{R} \) be differentiable, with \( f'(0) = 0 \). Define \( Q(x) = \frac{f(x) - f(0)}{x} \) \( (x \neq 0) \), \( Q(0) = 0 \). Suppose that \( f' \) is differentiable at 0, with \( f''(0) = 0 \). Show that \( Q \) is differentiable at 0, with \( Q'(0) = 0 \).

2. Let \( f : \mathbb{R} \to \mathbb{R} \) be differentiable at 0, with \( f(0) = 0 \). Let \( \{a_n\} \) be a real sequence such that \( \sum_n |a_n| < \infty \). Show that the series \( \sum_n f(a_n x) \) converges for every \( x \), and that the sum of this series is differentiable at 0.

3. a) Let \( \{a_n\} \) and \( \{b_n\} \) be bounded sequences in \( \mathbb{R} \). Prove that
   \[
   \liminf a_n + \limsup b_n \leq \limsup (a_n + b_n).
   \]
   b) Give an example where \( \liminf a_n + \limsup b_n < \limsup (a_n + b_n) \).

4. Let \( f \) be a continuous function on \( [a, b] \) with \( f(x) \geq 0 \) for all \( x \in [a, b] \). If \( \int_a^b f(x)dx = 0 \), prove that \( f(x) = 0 \) for all \( x \in [a, b] \).

5. The surface \( \{(x, y, z) \in \mathbb{R}^3 : xy + z = 1/2\} \) intersects the unit sphere in \( \mathbb{R}^3 \) in a curve \( C \). Show that for each point \( P \) in \( C \) there is a neighborhood of \( P, V_P \), and an open interval \( I \) in \( \mathbb{R} \) such that \( V_P \cap C \) is diffeomorphic to \( I \).

6. Let \( \{K_n\}_{n=1}^{\infty} \) be a decreasing sequence of nonempty closed subsets of a compact metric space \( X \). Show that if an open subset \( V \) of \( X \) contains the intersection of the \( K_n \)'s, then there exists \( N \) such that \( K_n \subseteq V \) for all \( n \geq N \).
Qualifying Examination in Analysis
14 January 2003

Work all six problems below.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$, and furthermore differentiable at $0$. Evaluate
\[
\frac{d}{dt} \int_a^b f(tx) \, dx \bigg|_{t=0}
\]
in terms of $a, b$ and $f'(0)$. Of course you must explain the steps in your calculation.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with $f(0) = 0$. Show that the series
\[
\sum_{n=0}^{\infty} f(2^{-n}x)
\]
converges for all $x$ in $\mathbb{R}$, and that the sum of the series is a continuous function of $x$.

3. For $y$ in $[0, +\infty)$, define $[y]$ to be $y$ minus the greatest integer $\leq y$, that is, the fractional part of $y$. Define $f : [0, +\infty) \to \mathbb{R}$ by
\[
f(x) = \sum_{n=1}^{\infty} \frac{[nx]}{2^n}
\]
Show that $f$ is Riemann integrable on every bounded subinterval of $[0, +\infty)$, and calculate $\int_0^1 f(x) \, dx$.

4. Carathéodory defines differentiability of a function at a point as follows. He calls $f : (a, b) \to \mathbb{R}$ differentiable at $x_0$ in $(a, b)$ provided there is a function $k : (a, b) \to \mathbb{R}$ which is continuous at $x_0$ and satisfies
\[
f(x) - f(x_0) = k(x)(x - x_0)
\]
for all $x$ in $(a, b)$.

(i) Show that this definition of differentiability is equivalent to the usual one.

(ii) Prove the chain rule using Carathéodory's definition.
5. Let $\phi : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Define $\rho_\phi$ on $C[a, b] \times C[a, b]$ by

$$\rho_\phi(f, g) := \int_a^b \phi(x)|f(x) - g(x)| \, dx.$$ 

(1) Show that $\rho_\phi$ is a metric on $C[a, b]$ if $\phi(x) > 0$ for all $x$ in $[a, b]$.

(2) Explain why the condition $\phi \geq 0$ is not enough to guarantee that $\rho_\phi$ is a metric on $C[a, b]$.

6. Let $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be defined by

$$\Phi(u, v) = (u + uv, u^2 + v^2, u^3 + 2v).$$

Show that there is a continuously differentiable $\mathbb{R}$-valued function $T$ defined in a neighborhood of $(0, 0, 0)$ in $\mathbb{R}^3$ such that for $(x, y, z)$ near $(0, 0, 0)$, one has $T(x, y, z) = 0$ if and only if $(x, y, z) = \Phi(u, v)$ for some $(u, v)$ near $(0, 0)$. In other words, the surface parametrized by $\Phi$ is a level surface near $\Phi(0, 0)$. (Hint: Consider $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $F(t, u, v) = t(x_0, y_0, z_0) + \Phi(u, v)$ for an appropriately chosen vector $(x_0, y_0, z_0)$.)
Instructions: Work ALL SIX problems. Justify your work.

1. Let \( \{x_n\} \) be a sequence of real numbers with the property that there is a number \( c, \ 0 < c < 1 \), such that
\[
|x_{n+1} - x_{n+2}| \leq c|x_n - x_{n+1}| \quad \forall n.
\]

Such a sequence is called contractive.

(i) Show that a contractive sequence is convergent.

(ii) Give an example of a convergent sequence that is not contractive.

(iii) If \( f \) is a differentiable function with the property that there is a number \( c < 1 \) such that \( |f'(x)| \leq c \) for all \( x \), show that for any given \( x_1 \) in \( \mathbb{R} \), the sequence \( \{x_1, x_2, \ldots\} \) defined recursively by \( x_{n+1} = f(x_n) \) is contractive.

2. Let \( \{K_n\} \) be a sequence of nonnegative Riemann integrable functions on \([0, 1]\) such that

(i) \( \int_0^1 K_n(x) \, dx = 1 \) for every \( n \).

(ii) \( K_n \to 0 \) uniformly on \([\delta, 1]\) for every \( \delta > 0 \).

Show that if \( f \) is Riemann integrable on \([0, 1]\) and continuous at 0, then
\[
\lim_{n \to \infty} \int_0^1 f(x)K_n(x) \, dx = f(0).
\]

3. Suppose that \( f \) is twice continuously differentiable on \([0, 1]\) with \( f(0) = f'(0) = 0 \) and \( |f''(x)| \leq 1 \) for all \( x \in [0, 1] \). Show that \( |f(1)| \leq 1/2 \) and that
\[
|\int_0^1 f(x) \, dx| \leq 1/6.
\]

4. Show that if \( h \) is a continuous real function on \([0, 1] \times [0, 1]\), then the function
\[
y \mapsto \int_0^1 h(x, y) \, dx
\]
is continuous on \([0, 1]\).

5. Let \( \{f_\alpha : \alpha \in A\} \) be a family of continuous nonnegative real functions on a compact metric space \( X \). Show that the pointwise infimum of the family, that is, the function \( f \) defined on \( X \) by \( f(x) = \inf_\alpha f_\alpha(x) \), attains a maximum value on \( X \).

6. Let \( f, g : \mathbb{R} \to \mathbb{R} \) be continuously differentiable with \( f(x), g(x), f'(x), g'(x) > 0 \) for every \( x \). Show that for all \( a, b \in \mathbb{R} \), the set
\[
\{(x, y) \in \mathbb{R}^2 : f(x) \cos y = a, \ g(x) \sin y = b\}
\]
has no cluster points.
Work all six problems – Justify your work.

1. Let \((X, d)\) be a metric space.
   a) Fix \(x_0 \in X\). For each \(x \in X\), define
   \[
   f_x(y) = d(x, y) - d(x_0, y), \quad \text{for } y \in X.
   \]
   Show that \(f_x\) is a bounded continuous function on \(X\).
   b) Let \(\|f\|_\infty = \sup \{|f(x)| : x \in X\}\). Show that \(\|f_x - f_y\|_\infty = d(x, y)\).

2. a) Show that if \(f\) is continuous on \((0, 1)\) and \(\lim_{x \to 0^+} f(x) = +\infty\), then \(f\) is not uniformly continuous. Argue from basic definitions.
   b) For which real values of \(\alpha\) is the function \(g_\alpha(x) = x^\alpha \ln(x)\) uniformly continuous on \((0, \infty)\)?

3. Let \(\{a_1, a_2, \ldots\}\) be a countable subset of \((0, 1)\), and let \(\sum_{j=1}^\infty |c_j| < \infty\). Define \(f\) on \([0, 1]\) by
   \[
   f(x) = \sum_{\{n : a_n < x\}} c_n
   \]
   (with the understanding that the sum over the empty set is 0). Show that \(f\) is Riemann integrable and express \(\int_0^1 f(x)dx\) as the sum of a series involving the \(a_j\)'s and \(c_j\)'s.

4. Prove Dirichlet's test for convergence of improper integrals: Let \(f\) be continuous and let \(g\) be \(C^1\) on \([a, \infty)\). Suppose that
   
   (i) the function \(F(x) = \int_a^x f(t) dt\) remains bounded as \(x \to \infty\);
   
   (ii) \(g'(x) \leq 0\) on \([a, \infty)\) and \(\lim_{x \to \infty} g(x) = 0\).

   Then \(\int_a^\infty f(x)g(x)dx\) converges.

5. a) Consider the surfaces \(S_1\) and \(S_2\) in \(\mathbb{R}^3\) defined respectively by the equations
   
   (1) \(x^2 - xy + yz^2 = 1\)
   
   (2) \(xz^2 + y^2 - 2yz = 0\)

   Notice that \((1, 1, 1) \in S_1 \cap S_2\). Show that there is a smooth injective map from an interval about 0 in \(\mathbb{R}\) to \(S_1 \cap S_2\) taking 0 to \((1, 1, 1)\).
   
   b) Why doesn't your argument work in the case of the two surfaces \(x^2 + y^2 + z^2 = 1\) and \(x^2 + 2y^2 + 3z^2 = 1\) and the point \((1, 0, 0)\)?

6. Let \(E\) be a nonempty subset of \(\mathbb{R}\). Suppose that \(y = \lim_n x_n = \lim_n y_n\), where each \(x_n\) belongs to \(E\) and each \(y_n\) is an upper bound for \(E\). Show that \(y = \sup E\).
Analysis Qualifying Exam - January 2002

Work all six problems.

1. Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be increasing on $I$. Suppose that $c \in \mathbb{R}$ is not an endpoint of $I$. Show that if there exists a sequence $\{x_n\}$ in $I$ with $x_n < c$ for $n$ odd and $x_n > c$ for $n$ even such that $c = \lim x_n$ and $f(c) = \lim f(x_n)$, then $f$ is continuous at $c$.

2. Let $(X, d)$ be a metric space, and let $K_1$ and $K_2$ be disjoint nonempty compact subsets of $X$. Show that there exist $k_1 \in K_1$ such that

$$d(k_1, k_2) = \inf \{d(x_1, x_2) : x_1 \in K_1, x_2 \in K_2\}.$$

3. Show that the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + t^2}$$

is a continuous function of $t$ on $\mathbb{R}$.

4. Let $F : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable, and let $\bar{a}$ in $\mathbb{R}^n$ be such that $\nabla F(\bar{a}) \neq 0$. Show that the level surface for $F$ through $\bar{a}$ can be smoothly parametrized near $\bar{a}$. In other words, show that there is a continuously differentiable $\mathbb{R}^n$-valued map $\phi$ defined in a neighborhood $U$ of the zero vector $\bar{0}_{n-1}$ in $\mathbb{R}^{n-1}$ such that $\phi(\bar{0}_{n-1}) = \bar{a}$, $\phi'(\bar{u})$ is one-to-one for every $\bar{u}$ in $U$, and $F \circ \phi$ is constant on $U$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$. Show that for any $a, b \in \mathbb{R}$ with $a < b$,

$$\lim_{n \to \infty} \int_{a}^{b} f(x + \frac{1}{n})dx = \int_{a}^{b} f(x)dx.$$

6. Define $f$ on $[-1, 1]$ by $f(x) = x$ if $x$ is rational, $f(x) = 0$ if $x$ is irrational. Find the upper and lower integrals

$$\overline{\int_{-1}^{1} f(x) \, dx} \quad \text{and} \quad \underline{\int_{-1}^{1} f(x) \, dx}$$

of $f$, that is, the infimum of the upper Riemann sums and the supremum of the lower Riemann sums.
Analysis Qualifying Exam – August 2001

Instructions: Work ALL SIX problems. Justify your work.

1. Let \( \{f_n\} \) be a sequence of continuous functions on \( \mathbb{R} \) which converges uniformly to a function \( f \) on a set \( A \). Show that

\[
\lim_{n \to \infty} f_n(x_n) = f(x)
\]

for every sequence of points \( x_n \in A \) such that \( x_n \to x \) and \( x \in A \).

2. Prove the following version of Taylor’s formula.

Let \( f : (0, 1) \to \mathbb{R} \). Assume that the second derivative of \( f \) exists on \( (0, 1) \). Then for each \( x \in (0, 1) \) there is a number \( c \) between \( x \) and \( \frac{1}{2} \) such that

\[
f(x) = f(1/2) + f'(1/2) \left( x - \frac{1}{2} \right) + \frac{f''(c)}{2} \left( x - \frac{1}{2} \right)^2.
\]

3. Let \( f, g : \mathbb{R}^n \to \mathbb{R} \) be continuously differentiable functions where \( n \geq 3 \). Assume that \( f(\vec{a}) = 0 = g(\vec{a}) \) and that \( \nabla f(\vec{a}) \) and \( \nabla g(\vec{a}) \) are linearly independent. Let \( \vec{v} \in \mathbb{R}^n \) be orthogonal to \( \nabla f(\vec{a}) \) and \( \nabla g(\vec{a}) \) for some \( \vec{a} \in \mathbb{R}^n \). Show that there is a continuously differentiable function \( \vec{x} \) from a real open interval about 0 to \( \mathbb{R}^n \) such that \( \vec{x}'(0) = \vec{v} \) and \( f(\vec{x}(t)) = 0 = g(\vec{x}(t)) \).

Hint: Solve

\[
\begin{cases}
  f(\vec{a} + r \nabla f(\vec{a}) + s \nabla g(\vec{a}) + t \vec{v}) &= 0 \\
  g(\vec{a} + r \nabla f(\vec{a}) + s \nabla g(\vec{a}) + t \vec{v}) &= 0
\end{cases}
\]

for \( r, s \) as a function of \( t \) near \((r, s, t) = (0, 0, 0)\).

4. Let \( f(x) = \frac{\sin(x)}{x} \) if \( x \neq 0 \) and \( f(x) = 1 \) if \( x = 0 \).

a) Show that \( \lim_{n \to \infty} \int_0^{\pi} f(x) dx \) exists and is finite.

b) Show that \( \lim_{n \to \infty} \int_0^{\pi} |f(x)| dx = \infty \).

5. Let \( f \) be a continuous function on \([a, b]\). Let \( \phi \) be continuous function on \( \mathbb{R} \). Define

\[
F(x) = \int_a^b f(y)\phi(x - y)dy.
\]

Show that \( F \) is well defined and continuous on \( \mathbb{R} \).

6. Let \( X \) be a metric space. Let \( K \) be a compact subset of \( X \) and \( p \) be a point in the complement of \( K \). Show that there are two open sets \( U \) and \( V \) in \( X \) such that \( K \subset U \), \( p \in V \) and \( U \cap V = \emptyset \).
Analysis Qualifying Exam - May 2001

Instructions: Work ALL SIX problems. Justify your work.

1. Let $f$ be a continuous real valued function on a compact set $E \subset \mathbb{R}^n$. Show that either $f$ has a zero or $f$ is bounded away from zero ($|f(x)| > m > 0$ for all $x \in E$ and some $m > 0$).

2. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = -1$ if $x \in [0, 1/2]$ and $f(x) = 1$ if $x \in (1/2, 1]$. Show that $f$ is Riemann integrable and that $\int_0^1 f(x) \, dx = 0$ using the definition of the Riemann integral.

3. Let $f : (a, b) \to \mathbb{R}$ be a differentiable function at $x_0 \in (a, b)$. Prove the following: Given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $u, v \in (a, b)$ satisfy $x_0 - \delta(\epsilon) < u \leq x_0 \leq v < x_0 + \delta(\epsilon)$, then we have

$$|f(v) - f(u) - (v - u)f'(x_0)| \leq \epsilon(v - u).$$

(Hint: subtract and add the term $f(x_0) - x_0f'(x_0)$.)

4. Show that the space of continuous functions on $[a, b]$, $C[a, b]$ with the sup-norm metric ($d(f, g) = \sup\{|f(x) - g(x)| : x \in [a, b]\}$) is a complete metric space.

5. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \frac{x^2 y^2}{x^2 + y^2}, \quad g(x, y) = \frac{x^2 y}{x^2 + y^2} \quad \text{if} (x, y) \neq (0, 0)$$

and $f(0, 0) = g(0, 0) = 0$. Is $f$ differentiable at $(0, 0)$? How about $g$? Justify your answers.

6. Let $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. For each $j \in \mathbb{N}$ define a function $f_j$ on $A$ by

$$f_j(0) = \sum_{k=0}^{\infty} \frac{k^j 2^j}{k! j!}, \quad f_j\left(\frac{1}{n}\right) = \sum_{k=0}^{n} \frac{k^j 2^j}{k! j!}.$$

(a) Show that $f_j$ is well-defined and continuous at $0 \in A$ for each $j \in \{0, 1, 2, 3, \ldots\}$.

(b) Let $f(x) = \sum_{j=0}^{\infty} f_j(x)$, $x \in A$. Show that $f$ is continuous at 0 in $A$. 

1
Work ALL the following problems

1. Let \( f \) be a continuous function with domain \([0, 1]\) and range \([0, 1]\). Show that there is an \( x_0 \in [0, 1] \) such that \( f(x_0) = x_0 \).

2. Let \( A \subset \mathbb{R} \) and suppose that \( f : A \to \mathbb{R} \) has the following property: for each \( \epsilon > 0 \) there exists a function \( g_\epsilon : A \to \mathbb{R} \) such that \( g_\epsilon \) is uniformly continuous on \( A \) and \( |f(x) - g_\epsilon(x)| < \epsilon \) for all \( x \in A \). Prove that \( f \) is uniformly continuous on \( A \).

3. Let \( \{x_n\} \) be a sequence of nonzero real numbers. Define the characteristic exponent of \( \{x_n\} \) by

\[
\chi(\{x_n\}) = \limsup_{n \to \infty} \frac{1}{n} \ln |x_n|.
\]

a. Give an example of a sequence \( \{x_n\} \) such that \( \chi(\{x_n\}) = 1 \) but \( \lim_{n \to \infty} \frac{1}{n} \ln |x_n| \) does not exist.

b. Show that \( \chi(\{x_n \cdot y_n\}) \leq \chi(\{x_n\}) + \chi(\{y_n\}) \).

4. A metric space \((S, d)\) is said to be complete if each Cauchy sequence in \( S \) converges to an element in \( S \). Let \( S \) be the space of continuous functions on \([0, 1]\) with the metric

\[
d(f, g) := \int_0^1 |f(x) - g(x)| dx.
\]

Show that \( S \) is not complete.

5. Let \( \sum_{n=1}^{\infty} a_n \) be a series of real numbers such that

\[
\lim_{n \to \infty} |a_n|^{\frac{1}{n}} = r < 1.
\]

Let \( S_n = \sum_{k=1}^{n} a_k \) and \( S = \sum_{k=1}^{\infty} a_k \). Prove that there exist \( r_1 \in (0, 1) \) and \( K > 0 \) such that

\[
|S - S_n| \leq \frac{r_1^{n+1}}{1 - r_1}, \quad \text{for } n \geq K.
\]

6. For \( E_1 \) and \( E_2 \) two sets in \( \mathbb{R} \) define

\[
E_1 + E_2 = \{x + y : x \in E_1, y \in E_2\}.
\]

a. Prove that if \( E_1 \) and \( E_2 \) are compact, then \( E_1 + E_2 \) is compact.

b. Give an example of a closed subset \( E \) of \( \mathbb{R} \) such that \( E + \mathbb{N} \), \( \mathbb{N} \) the set of natural numbers, is not closed.
QUALIFYING EXAM IN ANALYSIS
AUGUST 2000

Work ALL the following problems

1. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function and \( K \) a compact subset of \( \mathbb{R} \). Show that \( f(K) \) is compact. Is the preimage of \( K \), \( f^{-1}(K) \) compact?

2. Prove the extended mean value theorem for integrals: if \( f \) and \( g \) are continuous on \([a, b]\) and \( g(x) > 0 \) for all \( x \in [a, b] \) then there exists \( c \in (a, b) \) such that

\[
\int_a^b f(x)g(x)\,dx = f(c) \int_a^b g(x)\,dx.
\]

3. Let \( f \) be a Riemann integrable function on \([0, 1]\). Compute the following limits. If the limit does not exist, explain why.

   a. \( \lim_{n \to \infty} n^q \int_0^{1/n^p} f(x)\,dx, \quad p > q > 0. \)

   b. \( \lim_{n \to \infty} n^q \int_0^{1/n^p} xf(x)\,dx, \quad p > q/2 > 0. \)

4. State the implicit function theorem for the equation defined by a real function of two variables equal to zero. \((F(x, y) = 0.)\) Show that the equation

\[
x^3 + y^3 - 2xy = 0
\]

defines \( y \) as a function of \( x \), \( y = f(x) \), in a neighborhood of the point \((1, 1)\) and that the second derivative of \( f \), \((f'')\), is negative in a neighborhood of the point \((1, 1)\).

5. Give an example of a sequence of functions \( \{f_n\}_{n=0}^\infty, f_n : [-1, 1] \to \mathbb{R} \) such that

\[
\lim_{n \to \infty} \left[ \lim_{x \to 0} f_n(x) \right] \neq \lim_{x \to 0} \left[ \lim_{n \to \infty} f_n(x) \right].
\]

Could such sequence \( \{f_n\}_{n=0}^\infty \) be uniformly convergent on \([-1, 1]\)?

6. Define the oscillation of a real function at a point \( x \) as

\[
w_f(x) = \inf_{\delta > 0} \{ \sup \{ |f(y) - f(z)| : y, z \in (x - \delta, x + \delta) \} \}.
\]

Show that \( f \) is continuous at \( x \) if and only if \( w_f(x) = 0 \).
Qualifying Exam in Analysis
Thursday 13 January, 2000

Work all six problems below. Assume that all quantities, functions, sequences, and so forth that appear are real.

1. Show that if $f$ is continuous on $[a, b]$ and twice continuously differentiable on $(a, b)$, with $f(a) < 0 < f(b)$ and $f''(x) \neq 0$ for all $x$ in $(a, b)$, then $f$ has exactly one zero in $[a, b]$.

2. For what values of $p$ can you be sure that
   \[ \sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^p} \]
   is a continuously differentiable function of $x$? Explain.

3. Let $(X, d)$ be a metric space, and let $f$ be a (real-valued) function on $X$.
   (a) Say what it means for $f$ to be continuous on $X$.
   (b) Say what it means for $f$ to be uniformly continuous on $X$.
   (c) Show that if $X$ is compact, then every continuous function on $X$ is uniformly continuous.

4. Show that if $f$ is Riemann-integrable on $[0, 1]$ and continuous at $0$, then
   \[ \lim_{t \to \infty} t \int_{0}^{1} \frac{f(x)}{(1 + tx)^2} dx = f(0). \]
   (Hint: $f_0^1 = f_0^0 + f_0^1$; $f(x) = f(x) - f(0) + f(0)$)

5. Show that if \{a_n\} and \{b_n\} are bounded sequences, then
   \[ \lim sup a_n + b_n \leq \lim sup a_n + \lim sup b_n. \]

6. Show that there is a $\delta > 0$ and a continuously differentiable function $f$ on the interval $(-\delta, \delta)$ such that
   \[ xf(x)^3 + f(x) = 1 \]
   for $|x| < \delta$. 
Qualifying Exam in Analysis
August 12, 1999

Work all six problems below.

1. Let \( \{x_1, x_2, \ldots\} \) be a convergent real sequence with limit \( x \). Let \( \{a_n\} \) be the corresponding sequence of averages, that is \( a_n = (x_1 + \ldots + x_n)/n \) for each \( n \). Show that \( \lim_n a_n = x \).

2. Give an example of a uniformly convergent sequence \( \{f_n\} \) of continuously differentiable functions on \([0,1]\) such that the sequence \( \{f'_n\} \) of derivatives fails to converge pointwise on \([0,1]\).

3. Let \( f \) be a continuously differentiable real function defined on an open ball in \( R^k \) about the vector \( \bar{a} \) in \( R^k \). Show that if \( \nabla f(\bar{a}) \) is nonzero, then for every vector \( \bar{v} \) in \( R^k \) orthogonal to \( \nabla f(\bar{a}) \) there is a continuously differentiable \( R^k \)-valued function \( \theta \) defined on an open interval about \( 0 \) in \( R \) such that \( \theta(0) = \bar{a} \), \( \theta'(0) = \bar{v} \), and the composition \( f \circ \theta \) is constant. (Hint: Let \( \bar{w} = \nabla f(\bar{a}) \) and apply the implicit function theorem to the real function \( g \) defined in a neighborhood of \( (0,0) \) in \( R^2 \) by \( g(s,t) = f(\bar{a} + s\bar{v} + t\bar{w}).\) )

4. Show that if the graph of a bounded real function on \( R \) is closed in \( R^2 \), then the function is continuous. (Remark: The assumption that the function be bounded is essential here.)

5. Let \( X \) be the set of all bounded real sequences \( \bar{x} = \{x_1, x_2, \ldots\} \) with metric \( d \) defined by

\[ d(\bar{x}, \bar{y}) = \sup_n |x_n - y_n|. \]

The subset \( E \) consisting of all sequences \( \bar{x} \) such that \( d(\bar{x}, \bar{0}) \leq 1 \) is plainly a closed subset of \( X \). Is \( E \) also compact? (Of course you must say why or why not.)

6. (a) Give an example of a continuous real function on \( R \) that is differentiable everywhere except at \( 0 \).

(b) Show that if \( f \) is a continuous real function on \( R \) differentiable on \( R \setminus \{0\} \) such that \( \lim_{x \to 0} f'(x) \) exists, then \( f \) is differentiable at \( 0 \) as well.
Qualifying Exam in Analysis
January 13, 1999

1. Show that if the real sequence \( \{x_n\} \) satisfies

\[
\limsup_n \frac{|x_{n+1} - x_n|}{|x_n - x_{n-1}|} < 1,
\]

then \( \{x_n\} \) converges.

2. Find the infimum, over all positive integers \( n \) and all partitions

\[
0 = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = 1
\]

of the interval \([0,1]\), of the numbers \( \sum_{j=1}^{n} (x_j^4 - x_{j-1}x_j^3) \).

3. Let \( f \) be a continuously differentiable real function on \([1, +\infty)\) such that \( |f'(x)| \leq x^{-2} \) for all \( x \geq 1 \). Show that \( f(x) \) has a (finite) limit as \( x \to +\infty \).

4. Given a bounded real sequence \( \{a_n\} \), for what values of the real exponent \( p \) can you be sure that the series

\[
\sum_{n=1}^{\infty} \frac{a_n}{n^p} \sin(nx)
\]

sums to a twice-continuously differentiable function of \( x \)? Explain your answer.

5. Let \( f \) be a real function on a compact metric space \( X \) with the property that \( \{x \in X : f(x) < r\} \) is open for every real number \( r \). Show that \( f \) has a maximum value on \( X \), that is, there exists \( x_0 \in X \) such that \( f(x) \leq f(x_0) \) for every \( x \in X \).

6. Define \( f : \mathbb{R}^2 \to \mathbb{R} \) by \( f(x,y) = x^3(x^2 + y^2)^{-1} \), with \( f(0,0) = 0 \). Show that \( f \) is continuous at \((0,0)\), and that the first-order partial derivatives \( f_x \) and \( f_y \) are defined at \((0,0)\), but that \( f \) is not differentiable at \((0,0)\). (Hint: Try using the chain rule to calculate the derivative with respect to \( t \) of \( f(t,t) \) at \( t = 0 \).)
Sample Qualifying Exam in Analysis

1. Show that the function \( x \mapsto \sqrt{x} \) is uniformly continuous on \([0, +\infty)\), but that the function \( x \mapsto x^2 \) is not.

2. (a) Let \( \{x_n\} \) be a sequence in a metric space \((X, d)\). Show that the set of subsequential limits of \( \{x_n\} \) is a closed subset of \(X\).

(b) For a bounded real sequence \( \{x_n\} \), recall that

\[
\limsup_n x_n = \lim_{n \to \infty} \left( \sup_{k \geq n} x_k \right).
\]

Use this definition to show that \( \limsup_n x_n \) is the largest subsequential limit of the sequence \( \{x_n\} \).

3. Show that if \( f \) is a continuously differentiable real function on the real interval \([a, b]\), then the difference quotients for \( f \) converge uniformly to \( f' \) in the following sense: for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[
\left| \frac{f(s) - f(t)}{s - t} - f'(t) \right| < \varepsilon
\]

whenever \( s, t \in [a, b] \) with \( 0 < |s - t| < \delta \).

4. Give an example of a sequence of continuous real functions on \([0,1]\) converging pointwise to 0 whose integrals over \([0,1]\) don't converge to 0.

5. Consider the function \( f : \mathbb{R}^3 \to \mathbb{R}^2 \) defined by

\[
f(x, y, z) = (x^2 + yz + z^3 - 1, 2xz + y - 1),
\]

and the set \( C = \{(x, y, z) : f(x, y, z) = (0, 0)\} \). It so happens that \( C \) consists of two smooth curves intersecting in a single point. What system of polynomial equations in \( x, y, z \) would you solve in order to find this point? (You needn't actually solve the system, though that is not particularly difficult.)

6. Find the supremum, over all positive integers \( n \) and all partitions

\[
1 = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = 2
\]

of the interval \([1,2]\), of the numbers

\[
n - \sum_{j=1}^{n} \frac{x_{j-1}}{x_j}
\]