ALGEBRA QUALIFYING EXAMINATION : AUGUST 2009

You must show all work to receive full credit. When in doubt, it is better to show more work than less work. All problems are of equal value.

1. Prove that, up to isomorphism, there is only one finite group of order 77.

2. Let \( R \) be a commutative ring. Recall that the Jacobson radical of \( R \) is the intersection of all of the maximal ideals of \( R \). (a) Prove that the Jacobson radical of \( R \) equals the set 
\[ \{ a \in R \mid 1 + ax \text{ is a unit for all } x \in R \} \]
You may use (without proof) the fact that any proper ideal of \( R \) is contained in a maximal ideal. (b) Find, with proof, the Jacobson radical of the polynomial ring \( \mathbb{Z}[X] \) in one variable over \( \mathbb{Z} \).

3. Let \( F \) be the smallest subfield of the complex numbers containing all of the fifth roots of 2. Find, with proof, a basis for \( F \) over \( \mathbb{Q} \).

4. Let \( A \) be an \( n \times n \) matrix over the field \( F \) such that the rank of \( A \) is \( r > 0 \). Prove that there exist invertible matrices \( P, Q \) with entries in \( F \) such that \( P A Q \) has \( r \) 1’s down the diagonal and zeroes elsewhere.

5. Let \( f(x) = (x^2 + 1) \cdot (x^2 + x + 1) \in \mathbb{R}[x] \).
   (i) Prove that there is no \( 5 \times 5 \) matrix with entries in \( \mathbb{R} \) that has \( f(x) \) as its minimal polynomial.
   (ii) Find two non-similar \( 6 \times 6 \) matrices over \( \mathbb{R} \) that have \( f(x) \) as their minimal polynomial. (Recall that matrices \( A \) and \( B \) are similar if there exists an invertible matrix \( U \) such that \( B = U A U^{-1} \).)
   (iii) Find the Jordan canonical forms for your matrices in (ii).

6. Let \( V \) be the set of complex numbers regarded as a vector space over \( \mathbb{R} \). The identity \( < z_1, z_2 > := Re(z_1 \cdot \overline{z_2}) \) defines an inner product on \( V \). For each \( w \in V \), \( T_w(z) := w \cdot z \), for all \( z \in V \), defines a linear operator on \( V \).
   (i) What is the determinant of \( T_w \) ?
   (ii) For which complex numbers \( w \) is \( T_w \) self-adjoint ?
   (iii) For which complex numbers \( w \) is \( T_w \) a unitary operator ?
   (iv) Find a unitary operator on \( V \) that is not \( T_w \) for any \( w \in V \).