1) Let \( \{x_n\} \) be a sequence of real numbers and let \( \{x_{n_k}\} \) be a subsequence converging to a real number \( x \). Show that 
\[
\lim_{n \to \infty} \inf n \leq x \leq \lim_{n \to \infty} \sup x.
\]

2) Let \( f : [0, +\infty) \to \mathbb{R} \) be a continuous function such that \( \lim_{t \to +\infty} f(t) = 0 \). Prove that 
\[
\lim_{t \to +\infty} e^{-t} \int_0^t e^{s} f(s) ds = 0.
\]

3) (Intermediate Value Theorem for Derivatives.) Suppose that \( f \) is differentiable on \( [a, b] \) with \( f'(a) \neq f'(b) \). Prove that if \( y_0 \) is a real number that lies between \( f'(a) \) and \( f'(b) \), then there is an \( x_0 \) in \( (a, b) \) such that \( f'(x_0) = y_0 \). (Note that \( f' \) is not assumed to be continuous.)

4) Let \( f : [a, b] \to \mathbb{R} \) be a Riemann integrable function on \( [a, b] \) and let \( a < c < b \). Show that \( f \) is Riemann integrable on \( [a, c] \) and \( [c, b] \) and that 
\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.
\]
(Hint: Recall that if \( P \) and \( Q \) are partitions of an interval and \( Q \) is a refinement of \( P \) then \( U(f, Q) - L(f, Q) \leq U(f, P) - L(f, P) \), where \( U \) and \( L \) are the usual upper and lower Riemann sums associated to the partitions.)

5) Let \( (X, d) \) be a metric space. Show that every compact set in \( X \) is closed. (Note: You cannot solve this problem by quoting a theorem that has the given statement of the problem as a special case.)

6) Prove that there exist functions \( u, v : \mathbb{R}^4 \to \mathbb{R} \), continuously differentiable on some ball \( B \) in \( \mathbb{R}^4 \) centered at the point \( (x, y, z, w) = (2, 1, -1, -2) \), such that \( u(2, 1, -1, -2) = 4 \), \( v(2, 1, -1, -2) = 3 \), and the equations 
\[
u^2 + v^2 + w^2 = 29
\]
and 
\[
\frac{u^2}{x^2} + \frac{v^2}{y^2} + \frac{w^2}{z^2} = 17
\]
both hold for all \( (x, y, z, w) \) in \( B \).