1) Let \( f, g \) be differentiable function on \( \mathbb{R} \) such that \( f(x_0) = g(x_0) \) and \( f(x) \leq g(x) \) for all \( x \) in an open interval containing \( x_0 \). Show that \( f'(x_0) = g'(x_0) \).

2) Let \( f : [a, b] \rightarrow \mathbb{R} \) be continuously differentiable. Show that for all \( c \) in \( [a, b] \), we have
\[
|f(c)| \leq \frac{1}{b-a} \int_a^b |f(x)| \, dx + \int_a^b |f'(x)| \, dx.
\]

3) For \( 1 < s < \infty \), consider the Riemann zeta function
\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.
\]
Show that
\[
\zeta(s) = s \int_1^\infty \frac{[x]}{x^{s+1}} \, dx,
\]
where \([x]\) denotes the greatest integer less than or equal to \( x \).

4) Consider the sequences \( \{f_n\} \) and \( \{f'_n\} \), where \( f_n(x) = \frac{1}{n} \exp(-n^2x^2) \) on the interval \([-1, 1]\). Show whether \( \{f_n\} \) converges pointwise, uniformly, or if it diverges at some point. Do the same with \( \{f'_n\} \). Justify your answers.

5) Show that if \( K \) is a compact set in \( \mathbb{R}^n \), and \( U \) is an open set such that \( K \subset U \), then there exist \( r > 0 \) and a finite collection of disjoint balls \( \{B(x_j, r)\}_{j=1}^N \) such that \( K \subset \bigcup_{j=1}^N B(x_j, 3r) \subset U \).

6) Consider the function \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(x, y) = xy^2 \sin(x^{-1}) \) if \( x \neq 0 \) and \( f(0, y) = 0 \). Show that \( f_x \) and \( f_y \) exist for all \( (x, y) \). Is \( f \in C^1(\mathbb{R} \times \mathbb{R}) \)? Is \( f \) differentiable at every point of \( \mathbb{R} \times \mathbb{R} \)? Justify your answers.