Instructions: Show your work. You may use a calculator but neither books nor notes.

1. (a) Give the definition of the condition number in the absolute and relative senses for evaluating a function $y = f(x)$.

(b) Determine if the problem of evaluating function $f = \ln(x)$ for largely positive $x$ is ill-conditioned or well-conditioned in each of the absolute and relative senses.

(c) Explain why $\sqrt{\frac{1}{1+\epsilon}} - 1$ is not a good way to compute it in floating-point computation for very small $\epsilon$. Describe a better way to compute the value.

2. Suppose that function $g(x) \in C^1[a, b]$ satisfies

$$a \leq g(x) \leq b, \quad \forall x \in [a, b]$$

and

$$|g'(x)| \leq \lambda, \quad \forall x \in [a, b]$$

where $\lambda \in [0, 1)$ is a constant.

(a) Prove that $g(x)$ has a unique fixed point $\alpha \in [a, b]$. You CANNOT use the contraction mapping theorem.

(b) Show that sequence $\{x_n\}$ generated by the fixed point iteration $x_{n+1} = g(x_n)$ converges to $\alpha$ for any $x_0 \in [a, b]$ and has the property

$$|\alpha - x_n| \leq \frac{\lambda^n}{1-\lambda}|x_0 - x_1|.$$

(c) Show that sequence $\{x_n\}$ defined in (b) satisfies

$$\lim_{n \to \infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha).$$

(d) Determine an interval $[a, b]$ on which conditions (1) and (2) are satisfied by function $g(x) = \sqrt{x + 2}$. Justify.
3. Consider the quadrature rule in the form

\[ \int_{-1}^{1} f(x) = w_0 f(x_0) + E(f), \]  

(3)

where \( E(f) \) is the error term.

(a) Find \( w_0 \) and \( x_0 \) such that (3) has the highest algebraic degree of precision. What is the algebraic degree of precision?

(b) Find the error term \( E(f) \) in (3).

(c) Assume that the interval \([a, b]\) is divided into \( n \) subintervals of equal length with nodes \( x_i = a + ih, \ i = 0, ..., n \) and \( h = (b - a)/n \). Derive the composite quadrature rule for integral \( f_a^b f(x) dx \) by applying (3) (with found \( w_0 \) and \( x_0 \)) to each of the subintervals.

(d) Find the error term for the composition rule. Your result should be simplified into a form involving no summation.

4. Suppose that \( A \in \mathbb{C}^{m \times m} \) is strongly diagonally dominant (by row), i.e.,

\[ |a_{ii}| > d_i := \sum_{j=1, j \neq i}^{m} |a_{ij}|, \quad i = 1, \ldots, m. \]

(a) Prove that all of the eigenvalues of \( A \) are nonzero. You CANNOT use Gerschgorin’s Theorem.

(b) Prove that if further \( A \) is Hermitian, then

\[ \kappa_2(A) := \|A\|_2 \|A^{-1}\|_2 \leq \frac{\max_{1 \leq i \leq m} (|a_{ii}| + d_i)}{\min_{1 \leq i \leq m} (|a_{ii}| - d_i)}. \]
5. Let \( A = U \Sigma V^* \) be a singular value decomposition (SVD) for a matrix \( A \in \mathbb{C}^{m \times m} \), where \( U, V \in \mathbb{C}^{m \times m} \) are unitary and \( \Sigma \) is nonnegative diagonal.

(a) Prove the matrix
\[
W = \frac{\sqrt{2}}{2} \begin{bmatrix} U & U \\ V & -V \end{bmatrix} \in \mathbb{C}^{2m \times 2m}
\]
is unitary.

(b) Define the Hermitian matrix
\[
B = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \in \mathbb{C}^{2m \times 2m}.
\]
Show that \( \|B\|_2 = \|A\|_2 \) and the Schur form of \( B \) is given by
\[
B = W \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix} W^*.
\]

(c) Let \( \tilde{\sigma} \) be a singular value of \( \tilde{A} := A + \delta A \) for a given perturbation \( \delta A \in \mathbb{C}^{m \times m} \). Prove that there exists a singular value \( \sigma \) of \( A \) such that
\[
|\tilde{\sigma} - \sigma| \leq \|\delta A\|_2.
\]

Hint: Let \( \tilde{B} \) be the double-sized matrix corresponding to \( \tilde{A} \) as \( B \) corresponding to \( A \) given in (b). Express \( \tilde{B} = B + \delta B \). Then apply the Bauer-Fike Theorem.

6. Consider the implicit linear multi-step method
\[
y_{k+1} = y_{k-1} + \frac{h}{3} \left( f(x_{k+1}, y_{k+1}) + 4f(x_k, y_k) + f(x_{k-1}, y_{k-1}) \right)
\]
for the numerical solution of the IVP
\[
y' = f(x, y), \quad y(x_0) = y_0.
\]

(a) Find the order of the local truncation error, i.e., find an integer \( p \) for which the local truncation error is \( O(h^p) \).

(b) Determine if this method is weakly stable. Is it strongly stable?

(c) Determine if this method is consistent with the underlying differential equation.

(d) Determine if this method is convergent.

(e) Can this method be recommended for use on general ordinary differential equations? Explain.