Qualifying Exam in Analysis  
August 2011

Instructions: Work ALL the following six problems. Write clearly on one side of your paper and write your name on every sheet that you use.

1) Let $f : [0, \infty) \to \mathbb{R}$ be an uniformly continuous function, such that the limit
$$\lim_{b \to \infty} \int_0^b f(x) \, dx$$
exists as a finite number. Prove that $\lim_{x \to \infty} f(x) = 0$. In addition, show (by means of a counterexample) that there is a function $g$, so that the improper Riemann integral $\int_0^\infty g(x) \, dx < \infty$, while $\lim_{x \to \infty} g(x)$ does not exist.

2) Let $f : [a,b] \to \mathbb{R}$ be differentiable in $(a,b)$ and continuous in $[a,b]$. Assume $f(a) > 0, f(b) > 0$.

Prove that
a) If $f$ has exactly one root $c$ in $(a,b)$, then $c$ is a double root (i.e. $f(c) = f'(c) = 0$),

b) If $f$ has an odd number of roots in $(a,b)$, then at least one of them is a double root.

3) Let $(X,d)$ be a compact metric space. We say that a function $f : X \to \mathbb{R}$ is lower semi-continuous, if for all $x_0 \in X$, $\liminf_{x \to x_0} f(x) \geq f(x_0)$. Prove that $\inf_{x \in X} f(x)$ is achieved, that is, there exists $x_0 \in X$, so that $f(x_0) \leq f(x)$ for all $x \in X$.

4) Consider the system of equations
$$\cos w + \sin x + \tan y = z + 1; \quad e^w + x = y + e^{-z}.$$
Show that near $(x,y,w,z) = (0,0,0,0)$, $(x,y)$ can be expressed as a differentiable function of $(w,z)$. Compute $\frac{\partial x}{\partial w}(0,0)$ and $\frac{\partial y}{\partial w}(0,0)$.

5) Let $\{a_n\}_n$ be a sequence of real numbers, $a_n \neq 0$. Define
$$\lambda_0 = \inf \{ \lambda : \text{there exists } C, \text{ such that } |a_n| \leq Ce^{n\lambda} \text{ for all } n \}$$
$$\mu_0 = \limsup_{n \to \infty} \frac{\ln |a_n|}{n}$$

Prove that $\lambda_0 = \mu_0$. You may assume that $\lambda_0, \mu_0$ are finite, although the equality holds in the case of infinite $\lambda_0, \mu_0$.

6) Let
$$F(x) = \sum_{n=0}^{\infty} a_n x^n,$$
where the power series converges in a neighborhood of the origin. Compute
$$\mu(F) = \sup \left\{ \delta > 0 : \text{there exists } \epsilon > 0 \text{ such that } \int_{-\epsilon}^{\epsilon} |F(x)|^{-\delta} \, dx < \infty \right\},$$
where the integrals are interpreted as improper Riemann integrals if $F(0) = 0$.

**Hint:** Consider the cases $F(0) = 0$ and $F(0) \neq 0$ separately.