ALGEBRA QUALIFYING EXAM: AUGUST 20, 2013

Show all work to receive full credit. When in doubt, it is better to show more work than less.

1. Let $G$ be a group with subgroups $H$ and $K$. Set $HK := \{ h \cdot k \mid h \in H \text{ and } k \in K \}$.

   (a) Prove that $HK$ is a subgroup if and only if $HK = KH$. Conclude that if either $H$ or $K$ is a normal subgroup of $G$, then $HK$ is automatically a subgroup of $G$. (8 Points)

   (b) For a finite subset $C$ of $G$, let $|C|$ denote the number of elements in $C$. Prove that if $H$ and $K$ are finite, then

   $|HK| = \frac{|H| \cdot |K|}{|H \cap K|}$ (8 points).

2. Let $R$ be an integral domain and $\{ P_n \}_{n=1}^{\infty}$ an infinite collection of prime ideals.

   (a) Prove that if $P_1 \supseteq P_2 \supseteq P_3 \supseteq \cdots$, then $\bigcap_{n=1}^{\infty} P_n$ is a prime ideal. (8 points)

   (b) Give an example to show that the conclusion of part (a) fails if the given collection of primes does not form a descending chain. (8 Points)

3. Let $F \subseteq K$ be fields and $f(X)$ be an irreducible polynomial with coefficients in $F$. Let $\alpha \in K$ be a root of $f(X)$.

   (a) Suppose that $\sigma : K \to K$ is an automorphism of $K$ fixing $F$. Prove that $\beta := \sigma(\alpha)$ is a root of $f(X)$. (6 points)

   (b) Prove that if $\beta \in K$ is also a root of $f(X)$, then there exists an isomorphism of fields $\tau : F(\alpha) \to F(\beta)$ that fixes $F$ and takes $\alpha$ to $\beta$. (10 points)

4. Let $f(X) := X^3 - 2 \in \mathbb{Q}[X]$.

   (a) Prove that $f(X)$ is irreducible over $\mathbb{Q}$. (3 points)

   (b) Describe the splitting field $K$ of $f(X)$ over $\mathbb{Q}$. (3 points)

   (c) Use problem 3 to prove that there exists an automorphism $\sigma : K \to K$ that cyclically permutes the roots of $f(X)$. (6 points)

   (d) For $\sigma$ in (c), describe the set $\{ \gamma \in K \mid \sigma(\gamma) = \gamma \}$. (Hint: the set of elements in question is a subfield of $K$.) (6 points)

5. Let $V$ be a finite dimensional vector space over a field and $T : V \to V$ a linear transformation. Show that there exists $n \geq 1$ such that $V = \text{kernel}(T^n) \bigoplus \text{image}(T^n)$. (16 points)

6. Recall that a square matrix $A$ over a field $F$ is said to be skew symmetric if $A^t = -A$. Let $V$ denote the space of $3 \times 3$ skew-symmetric matrices over $\mathbb{C}$ and $M_3(\mathbb{C})$ the space of all $3 \times 3$ matrices over $\mathbb{C}$. For a fixed matrix $A \in M_3(\mathbb{C})$, let $\phi_A : V \to V$ denote the linear transformation $\phi_A(U) = AU + U^t A^t$, for all $U \in V$.

   (a) Show that a matrix in $V$ is either nilpotent or has three distinct eigenvalues. (4 points)

   (b) Fix a basis $B$ for $V$. For $A \in M_3(\mathbb{C})$, let $\hat{A}$ denote the matrix of $\phi_A$ with respect to $B$. Prove that $\hat{A}^t = \hat{A}$, for all $A, B \in M_3(\mathbb{C})$. (6 points)

   (c) Suppose $A \in M_3(\mathbb{C})$ has three distinct eigenvalues. Write the eigenvalues of $\hat{A}$ in terms of the eigenvalues of $A$. (Hint: Part (c) should be helpful.) (8 points)