Instructions: Solve all problems. The problems are weighted equally. Show your work. You may use a graphing calculator but you cannot use books, notes, or other assistance.

1. Suppose that $N(h)$ is an approximation to $M$ for every $h > 0$ and

   $$ M - N(h) = K_1 h + K_2 h^2 + K_3 h^3 + \cdots, $$

where $K_1, K_2, K_3$ are nonzeros constants independent of $h$. Use $N(h)$, $N(h/2)$, and $N(h/3)$ to produce an $O(h^3)$ approximation to $M$.

2. Let $f(x) \in C^{n+1}[a, b]$. Suppose $P_n(x)$ is a polynomial of degree at most $n$ that interpolates $f(x)$ at the distinct nodes $x_0, x_1, \ldots, x_n \in [a, b]$.

   (a) Give the Lagrange interpolation polynomial $L_i$ and the Lagrange interpolation formula for $P_n(x)$.
   
   (b) Prove
   $$ \sum_{i=0}^{n} L_i(x) = 1 \quad \text{and} \quad f(x) - P_n(x) = \sum_{i=0}^{n} (f(x) - f(x_i)) L_i(x). $$
   
   (c) Prove that if $f$ is a polynomial of degree $k$, then for $n > k$, the divided difference
   $$ f[x_0, x_1, \cdots, x_n] = 0. $$

3. Let $I_n(f) = w_1 f(x_1) + \cdots + w_n f(x_n)$ be the quadrature to approximate $I(f) = \int_{a}^{b} f(x) dx$. Prove that for any choice of $n$ distinct real numbers $x_1, \cdots, x_n \in [a, b]$ and real numbers $w_1, \cdots, w_n$, the degree of precision of $I_n(f)$ is at most $2n - 1$. State the choice of $x_1, \cdots, x_n$ which reaches the maximum degree of precision without proof.

4. Consider the multistep method

   $$ y_n - y_{n-2} = h(f_n - 3f_{n-1} + 4f_{n-2}), $$

for the numerical solution of the initial value problem

   $$ y' = f(t, y), \quad y(t_0) = y_0. $$

   (a) Find the order of the local truncation error.
   
   (b) Determine if this method is weakly stable. Is it strongly stable?
   
   (c) Determine if this method is consistent with the underlying differential equation.
   
   (d) Determine if this method is convergent.

5. Let $A, E \in \mathbb{C}^{m \times m}$. Suppose that $\sigma_{\text{min}} > 0$ is the smallest singular value of $A$ and $\|E\|_2 < \sigma_{\text{min}}$.

   (a) Prove that $\|A^{-1}E\|_2 < 1$ and $A + E$ is invertible.
(b) Let $b \in \mathbb{C}^m$. Suppose that $x \in \mathbb{C}^m$ is the solution to $Ax = b$ and $\tilde{x}$ is the solution to $(A + E)\tilde{x} = b$. Prove
\[
\frac{\|x - \tilde{x}\|_2}{\|x\|_2} \leq \frac{1}{1 - \|E\|_2/\sigma_{\text{min}}(A)} \frac{\|E\|_2}{\|A\|_2}.
\]

(c) Let $0 \neq b \in \mathbb{C}^m$ and $f \in \mathbb{C}^m$. Suppose $x \in \mathbb{C}^m$ is the solution to $Ax = b$ and $\tilde{x}$ is the solution to $A\tilde{x} = b + f$. Prove
\[
\frac{\|x - \tilde{x}\|_2}{\|x\|_2} \leq (\|A\|_2\|A^{-1}\|_2) \frac{\|f\|_2}{\|b\|_2}.
\]

6. Suppose $A \in \mathbb{C}^{m \times m}$ and $\tau$ is a scalar. Let $A - \tau I = QR$, where $Q$ is unitary and $R$ is upper triangular. Define $B = RQ + \tau I$.

(a) Prove $B = Q^*AQ$.

(b) Prove if $A$ is upper Hessenberg and is unreduced, e.g., $a_{i,j} = 0$ for all $i > j + 1$, and $a_{j+1,j} \neq 0$ for $j = 1, \ldots, m - 1$, then $B$ is also upper Hessenberg.
   
   \text{Hint:} Show $Q$ is upper Hessenberg by comparing the first $m - 1$ columns on both sides of $A - \tau I = QR$.

(c) Prove if $A$ is upper Hessenberg and is unreduced, and $\tau$ is an eigenvalue of $A$, then $e_m^TB = \tau e_m^T$, i.e., the last row of $B$ is $[0, \ldots, 0, \tau]$. 