ALGEBRA QUALIFYING EXAM: AUGUST 19, 2014

Show all work to receive full credit. When in doubt, it is better to show more work than less.

1. Let $W_1 \subseteq V_1$ and $W_2 \subseteq V_2$ be vector spaces over the field $F$. Set $U := \{ T \in \mathcal{L}(V_1, V_2) \mid T(W_1) \subseteq W_2 \}$, where $\mathcal{L}(V_1, V_2)$ denotes the space of linear transformations from $V_1$ to $V_2$. Note that $U$ is a subspace of $\mathcal{L}(V_1, V_2)$.

(i) Show that there exists a surjective linear transformation $\phi : U \to \mathcal{L}(V_1/W_1, V_2/W_2)$. (8 points)

(ii) Identify (with proof) the kernel of $\phi$. (4 points)

(iii) Assume that $V_1$ and $V_2$ are finite dimensional over $F$. Find a formula for the dimension of $U$. (4 points)

2. Let $V$ denote the vector space of $2 \times 2$ matrices over the field of complex numbers and set $A := \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$.

Let $T_A : V \to V$ be the linear transformation given by $T_A(B) = A \cdot B$, for all $B \in V$. Find the Jordan canonical form for $T_A$ and find a basis $B$ for $V$ such that the matrix of $T_A$ with respect to $B$ is in Jordan canonical form. (18 points)

3. Let $V$ be a finite dimensional vector space over the field $F$ and $T : V \to V$ a linear transformation.

(i) For $v \in V$, let $\mu_{T,v}(X)$ denote the monic polynomial of least degree such that $\mu_{T,v}(T)(v) = 0$. Prove that $v$ is a cyclic vector for $V$ with respect to $T$ if and only if the degree of $\mu_{T,v}(X)$ equals the dimension of $V$. (5 points)

(ii) Suppose $V = W_1 \oplus W_2$, for $T$-invariant subspaces $W_1, W_2 \subseteq V$. Write $\mu_1(X)$ for the minimal polynomial of $T|_{W_1}$ and $\mu_2(X)$ for the minimal polynomial of $T|_{W_2}$ and suppose that $\mu_1(X)$ and $\mu_2(X)$ are relatively prime. For $w_i \in W_i$, prove that $v := w_1 + w_2$ is a cyclic vector for $V$ with respect to $T$ if and only if $w_i$ is a cyclic vector for $W_i$ with respect to $T|_{W_i}$, for $i = 1, 2$. (8 points)

(iii) Give a specific example where the conclusion of (i) fails in case $\mu_1(X)$ and $\mu_2(X)$ are not relatively prime. (5 points)

4. Let $p > 5$ be a prime that is not congruent to 1 modulo 5. Prove that any group of order $15p$ contains a subgroup of order $5p$. State carefully any theorem you use to prove this result. (16 points)

5. Let $R$ be a principal ideal domain. For $f, g \in R$, show that $f^{1000}g^{1014}$ belongs to the ideal generated by $f^{2014}$ and $g^{2014}$. (16 points)

6. Show that $X^5 - 2$ is irreducible over the field $\mathbb{Z}_{31}$. (16 points)