Qualifying Exam in Analysis
August 2015

Instructions: Work ALL the following six problems. Write clearly on one side of your paper and write your name on every sheet that you use.

Remark: None of the problems require the use of results from measure theory or complex analysis. You cannot invoke results about measurable functions or analytic functions unless you prove them (such results are not needed to solve the problems). If you are in doubt about a statement ask the faculty proctoring the exam.

1) Let \( \{x_n\} \) and \( \{y_n\} \) be bounded sequences of non-negative real numbers. Prove that

\[
\limsup_{n \to \infty} (x_n y_n) \leq \left( \limsup_{n \to \infty} x_n \right) \left( \limsup_{n \to \infty} y_n \right).
\]

Show by example that this inequality can be strict.

2) Show that the series \( \sum_{n=1}^{\infty} \left( e^{\frac{x}{n}} - 1 - \frac{x}{n} \right) \) converges uniformly in \([-A, A]\) for any \( A > 0 \). Show also that the sum of the series is a function with derivatives of all orders in \( \mathbb{R} \).

3) Let \( f : \mathbb{R} \to \mathbb{R} \) be a function such that, \( f''(x) \) exists, \( f''(x) \geq 0 \) for every \( x \in \mathbb{R} \). Let \( \lambda \in (0, 1) \) and \( a < b \). Show that

\[
f(\lambda a + (1 - \lambda) b) \leq \lambda f(a) + (1 - \lambda) f(b).
\]

(Hint: Show first that there exist \( c \leq d \) such that \( f(\lambda a + (1 - \lambda) b) = f(a) + (1 - \lambda)(b - a)f'(c) \) and that \( f(\lambda a + (1 - \lambda) b) = f(b) - \lambda(b - a)f'(d) \).)

4) Given a continuous function \( f : \mathbb{R} \to \mathbb{R} \), define the sequence

\[
f_n(x) = \frac{n}{2} \int_{x - \frac{1}{n}}^{x + \frac{1}{n}} f(t) dt.
\]

(i) Show that the sequence \( \{f_n\}_{n=1}^{\infty} \) converges uniformly to \( f \) on every finite closed interval \([a, b]\).

(ii) Show that the sequence \( \{f_n\}_{n=1}^{\infty} \) may not converge uniformly in all \( \mathbb{R} \).

5) Let \((X, d)\) be a compact metric space. Let \( f : X \to X \) be a function such that \( d(f(x), f(y)) = d(x, y) \) for all \( x, y \in X \). Prove that \( f \) is a bijection from \( X \) onto \( X \).

(Hint: Assume that there exist \( x_0 \in X \setminus f(X) \). Recursively define the sequence \( x_{n+1} = f(x_n), n \geq 0 \) and show that there exist \( \epsilon > 0 \) such that \( d(x_n, x_m) > \epsilon \) for all \( m > n \geq 0 \).)

6) Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a function defined by

\[
f(x, y) = \begin{cases} 
x^2 + y^2 & \text{if } x \text{ and } y \text{ are rational} \\
0 & \text{otherwise}
\end{cases}
\]

Find all the points \((x, y) \in \mathbb{R}^2 \) (if any) where \( f \) is differentiable. Justify your answer.