1. An urn initially contains 5 white and 7 black balls. Each time a ball is selected, its color is noted and it is replaced in the urn along with 2 other balls of the same color. Compute the probability that the first 2 balls selected are black given that the next 2 are white.

2. Let \((X, Y)\) be a random vector with uniform distribution on the interior of the polygon with vertices \((2, 0), (-2, 0), (0, 1)\) and \((0, -1)\).
   a) Find the density of the random variable \(X\).
   b) Find the probability \(P(X^2 < 2Y)\).
   c) Find the distribution of the random vector \((X + 2Y, X - 2Y)\). Are the random variables \(X + 2Y\) and \(X - 2Y\) independent?

3. Let \(\{X_n\}_{n \geq 1}\) be a sequence of independent random variables such that, for each \(n \geq 1\), \(X_n\) has an exponential distribution with parameter \(\lambda_n > 0\).
   a) Assuming \(\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \lambda_k = \lambda > 0\), show that the sequence
      \[Y_n = n \min(X_1, X_2, \ldots, X_n)\]
   converges in distribution and find the limit.
   b) Assuming \(\lambda_n = \lambda\) for all \(n \geq 1\), show that the sequence
      \[Z_n = \frac{X_1X_2 + X_2X_3 + \cdots + X_{n-1}X_n}{n}\]
   converges in \(L^2\) and find the limit.

4. Let \(\{X_n\}_{n \geq 1}\) and \(\{Y_n\}_{n \geq 1}\) be two sequences of random variables such that, for each \(n \geq 1\), the random variables \(X_n\) and \(Y_n\) are independent. Suppose that \(X_n\) converges in distribution to the normal law \(N(2, 3)\) and \(Y_n\) converges in distribution to the normal law \(N(-3, 4)\). Show that \(X_n + Y_n\) converges in distribution and compute the limit.
Statistics

1. Let \( X_1, \ldots, X_n \) be a random sample from a Poisson distribution with parameter \( \theta > 0 \). Find the minimum variance unbiased estimator of \( \theta^2 e^{-\theta} (1 - \theta/3) \). Hint: the previous function of the parameter can be expressed using the sampling pmf.

2. Consider a random sample from a bivariate normal distribution with known covariance matrix. The parameter vector is the pair of the expected values of the normal random variables. Check that pair of the corresponding sample means is an efficient estimator of the parameter vector.

3. Consider a random sample from a Binom(\( m, \theta \)) distribution, where \( \theta \in (0, 1) \) is a parameter and \( m \) is a known positive integer. Let the prior distribution of the parameter be Beta(\( \alpha, \beta \)), where \( \alpha > 0 \) and \( \beta > 0 \) are known.

   (a) Find the Bayes estimator of \( \theta \).

   (b) Find the Bayes estimator of \( \theta^2 \).

4. Let \( X \) be an integer-valued random variable whose pmf is either \( g(x) = \frac{1}{13}; x = 0, 1, \ldots, 12 \), or

\[
h(x) = \begin{cases} 
\frac{x}{36} & \text{if } x = 0, 1, \ldots, 6 \\
\frac{1}{3} - \frac{x}{36} & \text{if } x = 7, 8, \ldots, 12 
\end{cases}
\]

On the basis of one observation \( X \), construct the most powerful test of \( H_0 : f_X = g \) vs. \( H_1 : f_X = h \) at a 0.25 level of significance. Find the power of the test.