1. Let $G$ be a finite group with $|G| = p^n$, $p$ prime.
   (i) Prove that $Z(G)$, the center of $G$, is non-trivial.
   (ii) Prove that if $N \subseteq G$ is a normal subgroup of order $p$, then $N \subseteq Z(G)$. Hint: Let $G$ act on $N$.
   (iii) Give an example of a non-abelian group group of order $p^n$ whose center contains more than one normal subgroup of order $p$.

2. Let $R$ be an integral domain. Suppose there exists a nonzero, non-unit $a \in R$ such that for every $r \in R$, there exists a unit $u$ and $n \geq 0$ such that $r = ua^n$. Such a ring is called a discrete valuation ring. Set $P := aR$. Let $R[\frac{1}{a}]$ denote the ring of polynomial expressions in $\frac{1}{a}$ with coefficients in $R$. Note, $\frac{1}{a} \notin R$.
   (i) Prove that $P$ is a maximal ideal, and in fact, the only maximal ideal.
   (ii) Prove that $\bigcap_{n=1}^{\infty} P^n = (0)$.
   (iii) Prove that $R[\frac{1}{a}]$ is a field.
   (iv) Let $R$ be the ring of formal power series over $\mathbb{Q}$. Thus, a typical element in $R$ is of the form $\sum_{i=0}^{\infty} \alpha_i x^i$, with $\alpha_i \in \mathbb{Q}$ and with addition and multiplication given just like for polynomials. Prove that $R$ is a discrete valuation ring.

3. Let $\mathbb{Z}^n$ denote the free abelian group of rank $n$, its elements being row vectors of length $n$. Let $A$ be an $r \times n$ matrix over $\mathbb{Z}$ and write $K_A$ for the subgroup of $\mathbb{Z}^n$ generated by the rows of $A$.
   (i) Suppose $B := PAQ$, where $P$ is an $r \times r$ invertible matrix over $\mathbb{Z}$ and $Q$ is an invertible $n \times n$ matrix over $\mathbb{Z}$. Prove that $\mathbb{Z}^n/K_A$ and $\mathbb{Z}^n/K_B$ are isomorphic as abelian groups.
   (ii) Suppose $A = \begin{pmatrix} 4 & -2 & 4 \\ 2 & 4 & 4 \end{pmatrix}$. Write $\mathbb{Z}^3/K_A$ as a direct sum of cyclic groups.

4. Let $f(x) := x^3 - 9x + 3 \in \mathbb{Q}[x]$ and $\alpha$ a root of $f(x)$.
   (i) Prove that $f(x)$ is irreducible over $\mathbb{Q}$.
   (ii) In the field $\mathbb{Q}(\alpha)$, write $(3\alpha^2 + 2\alpha + 1)^{-1}$ in terms of the basis $1, \alpha, \alpha^2$.

5. Let $A$ be an $n \times n$ matrix over the field $F$ and $F^n$ denote the vector space of column vectors of length $n$. Suppose $A$ is idempotent, i.e., $A^2 = A$.
   (i) Prove that $F^n = U \oplus W$, where $A \cdot u = 0$, for all $u \in U$ and $A \cdot w = w$, for all $w \in W$.
   (ii) If $F = \mathbb{Z}_p$, how many idempotent $3 \times 3$ matrices are there?

6. Find the characteristic polynomial, the minimal polynomial, the rational canonical form and the Jordan canonical form for the matrix $A = \begin{pmatrix} 2 & 0 & 0 \\ 9 & 7 & 5 \\ -9 & -5 & -3 \end{pmatrix}$. Here we assume $A$ has coefficients in a field of characteristic zero. How does your answer change if the entries of $A$ belong to a field of positive characteristic?