Numerical Analysis Qualifying Exam, January 2017

Solve all problems. The problems are weighted equally. Show your work.

**Problem 1.** The fixed point iterations $x_{n+1} = x_n + \lambda(x_n)f(x_n)$ are a frequently reinvented set of methods for “solving” the nonlinear equation $f(x) = 0$. Each choice of $\lambda(x)$ gives a different numerical method. Assume that $f(x)$ and $\lambda(x)$ have at least two continuous derivatives and that the roots of $f(x)$ have multiplicity one.

i. A simple (and common) choice is to take $\lambda(x)$ to be a constant function $\lambda(x) = \lambda_0$. For which values of $\lambda_0$ does the fixed point iteration converge linearly (or faster) to a root $x_*$ from initial conditions close enough $x_*$? Explain.

ii. For which values of $\lambda_0$ (if any) is the rate of convergence at least quadratic? Explain.

iii. Under what condition on $\lambda(x)$ does the fixed point iteration converge linearly (or faster) to a root $x_*$ from initial conditions close enough $x_*$? Explain.

iv. Under what condition on $\lambda(x)$ does the fixed point iteration converge quadratically (or faster) to a root $x_*$ from initial conditions close enough $x_*$?

**Problem 2.**

(i) Derive the two-point Gaussian quadrature formula with its associated error

$$
\int_{-1}^{1} f(x)dx = A_1 f(-x_1) + A_1 f(x_1) + E(f), \quad E(f) = cf^{(d+1)}(\xi).
$$

Find $A_1, x_1$ and $c, d$.

(ii) Find the one-point and two-point Gaussian quadrature formulas

$$
\int_{0}^{1} x f(x)dx \approx \sum_{j=1}^{n} w_j f(x_j).
$$
(iii) Lobatto’s rule is a Gaussian formula (weight function \( w(x) \equiv 1 \)) for integrating \( I(f) = \int_{-1}^{1} f(x) \, dx \) except that it includes \( \pm 1 \) as two fixed abscissas. It has the form

\[
I(f) \approx A_0 f(-1) + A_1 f(x_1) + \cdots + A_{k-1} f(x_{k-1}) + A_k f(1).
\]

Derive the Lobatto rule for the case \( k = 2 \) and show that it is exact for polynomials of degree \( \leq 3 \).

**Problem 3.** For \( A \in \mathbb{R}^{n \times n} \) and \( A = A^T \), Rayleigh Quotient Iteration is defined by:

- select \( \vec{v}^{(0)} \in \mathbb{R}^n \) such that \( \|\vec{v}^{(0)}\|_2 = 1 \)
- for \( k = 1, 2, \ldots \)
  - compute \( \lambda^{(k-1)} = [\vec{v}^{(k-1)}]^T A [\vec{v}^{(k-1)}] \)
  - solve \( (A - \lambda^{(k-1)} I) \vec{w} = \vec{v}^{(k-1)} \) for \( \vec{w} \)
  - compute \( \vec{v}^{(k)} = \vec{w}/\|\vec{w}\|_2 \).

Assume \( A = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \) with \( d_1 > d_2 \), and \( \vec{v}^{(k-1)} = [v_1^{(k-1)}, v_2^{(k-1)}]^T \) with \( v_1^{(k-1)} \neq 0 \) and \( v_2^{(k-1)} \neq 0 \).

(a) Express components of \( \vec{v}^{(k)} = [v_1^{(k)}, v_2^{(k)}]^T \) in terms of \( v_1^{(k-1)} \) and \( v_2^{(k-1)} \).

(b) Assume, in addition, that \( v_1^{(k-1)} \approx 1 \) and \( v_2^{(k-1)} \approx 0 \). Show that \( \|\vec{v}^{(k)} - \vec{e}_1\|_2 \approx \|\vec{v}^{(k-1)} - \vec{e}_1\|_2^3 \), where \( \vec{e}_1 = [1, 0]^T \). What is the significance of this result? **Hint:** \( \sqrt{1 + x} \approx 1 + x/2 \) for \( x \approx 0 \).

**Problem 4.**

i. State the Gerschgorin circle theorem. (HINT: Don’t forget to state both parts of the theorem.)

ii. Suppose that \( D \in \mathbb{R}^{n \times n} \) is diagonal and \( E \in \mathbb{R}^{n \times n} \) is any matrix. Use the Gerschgorin circle theorem to show that if

\[
\|E\|_\infty < \min_{i \neq j} \left| \frac{d_{ii} - d_{jj}}{2} \right|
\]

then there is an ordering of the eigenvalues of \( D + E \), \( \mu_1, \mu_2, \mu_3, \ldots, \mu_n \) such that

\[
|d_{ii} - \mu_i| < \|E\|_\infty
\]

for all \( i = 1, 2, 3, \ldots, n \). (HINT: First show that the \( i \)-th Gerschgorin disk is contained in the disk with center \( d_{ii} \) and radius \( \sum_{j=1}^{n} |e_{ij}| \).)
Problem 5. The following questions refer to numerical solutions of the initial value problem $y' = -y^2$, $y(0) = 1$ using a fixed step size $h$. If appropriate, use the following notation. For $x_j = x_0 + jh$, denote a particular method’s approximation to $y(x_j)$ by $y_j$ and denote $f(x_j, y_j)$ by $f_j$.

i. Define what is the local truncation error of a numerical method for “solving” the initial value problem.

ii. Define what is a convergent numerical method for solving the initial value problem. (Give a rigorously correct definition—don’t just state a theorem.)

iii. Define what is a consistent numerical method. (Give a rigorously correct definition—don’t just state a theorem.)

iv. (a) Write down the Taylor method with $O(h^3)$ local truncation error.

(b) Write down a Runge-Kutta method with local truncation error $O(h^3)$ or smaller.

(c) Write down a linear multi-step method with local truncation error $O(h^3)$ or smaller.