Directions: Work each problem on a separate sheet of paper. Each problem is worth the same number of points.

1. State a corollary to the Seifert-Van Kampen theorem where $U, V$, and $U \cap V$ are open, arcwise connected subsets of $X = U \cup V$, and $V$ is simply connected. Use this result to compute the fundamental group of the Klein bottle.

2. Let $\phi : (X, x_0) \to (Y, y_0)$ be continuous, and suppose $p : (\tilde{Y}, \tilde{y}_0) \to (Y, y_0)$ is a covering projection. Prove that there’s a map

$$\tilde{\phi} : (X, x_0) \to (\tilde{Y}, \tilde{y}_0)$$

with the property $p \tilde{\phi} = \phi$ if and only if

$$\phi_*(\pi(X, x_0)) \subseteq p_*(\pi(\tilde{Y}, \tilde{y}_0)).$$

You may assume that all spaces are arcwise connected. [You do not need to prove that $\tilde{\phi}$ is continuous, but make sure you prove that it’s well-defined.]

3. Prove that the homology sequence of the pair $(X, A)$

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{\partial} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

is exact at $H_n(X)$. 