Combinatorics and Graph Theory Prelim, August 19, 1994

PART I. Do Problem #1 and TWO out of the THREE Problems, #2, #3, #4.

1. (a) Give a combinatorial proof of \( \binom{n}{k} \binom{k}{m} \) \( \binom{n}{m} \binom{n-m}{k-m} \).

(b) Show that the number of partitions of \( n \) into distinct parts equals the number of partitions of \( n \) into odd parts.

(c) Let \( p_1, p_2, \ldots, p_k \) be positive integers, and let \( n = (\sum_{i=1}^{k} p_i) - k + 1 \). Prove: If \( f : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, k\} \), then for some \( i, 1 \leq i \leq k \), \( |f^{-1}(i)| \geq p_i \). (\( |f^{-1}(i)| \) is the cardinality of the inverse image of \( i \) under \( f \).)

(d) The lattice of partitions of \( \{1, 2, \ldots, n\} \) contains how many maximal chains from \( 12 \cdots n \) to \( 1-2-\cdots-n \)? (\( 12 \cdots n \) is the partition having all elements in one subset; \( 1-2-\cdots-n \) is the partition consisting of \( n \) singleton subsets.)

2. Let \( a_n \) be the number of sequences of length \( n \) of letters of the English alphabet (which contains 26 different letters) in which between them the five vowels A, E, I, O, U occur an even number of times.

(a) Show that the sequence \( \{a_n\} \) satisfies the recurrence relation \( a_{n+1} = 16a_n + 5(26^n) \) for \( n \geq 1 \) with \( a_1 = 21 \).

(b) Find the generating function for the sequence \( \{a_n\} \).

(c) Find a formula for \( a_n \).

3. (a) State Polya's Theorem.

(b) Describe all symmetries of a regular hexagon.

(c) How many essentially different colorings of a regular hexagon are there, where the vertices are colored from a fixed set of \( c \) colors?

4. Assuming only the definition of a projective plane, prove the following:
   If \( (P, L) \) is a projective plane \( (P = \text{ set of points}, \ L = \text{ set of lines}) \), then there exists an integer \( n \geq 2 \) such that each line contains exactly \( n + 1 \) points, each point is on exactly \( n + 1 \) lines, and \( |P| = |L| = n^2 + n + 1 \).
PART II. Do THREE of the following FOUR problems.

1. Prove: If $G$ is a graph with all vertices of even degree, then the edges of $G$ can be partitioned into the edge sets of cycles of $G$.

2. Prove: For any graph $G$,

$$
\kappa(G) \leq \kappa_1(G) \leq \delta(G).
$$

($\kappa(G)$ is the (vertex-)connectivity; $\kappa_1(G)$ is the edge-connectivity; and $\delta(G)$ is the minimum degree of $G$.)

3. Let $Q_n$ be the graph of the $n$-cube. ($Q_1 = K_2, Q_n = Q_{n-1} \times K_2$.)

Answer each of the following questions, and prove your answers are correct.

(a) For which $n$ is $Q_n$ Eulerian?

(b) For which $n$ is $Q_n$ Hamiltonian?

(c) For which $n$ is $Q_n$ planar?

(d) For which $n$ is $Q_n$ 1-factorable? (1-factorable means its edges can be partitioned into perfect matchings.)

4. An edge-dominating set for a graph $G$ is a set $F$ of edges such that every edge of $G$ belongs to $F$ or is adjacent to (shares a vertex with) some edge of $F$. The edge-dominating number $\sigma_1(G)$ is the minimum cardinality of an edge-dominating set in the graph $G$. The edge independence number $\beta_1(G)$ is the maximum cardinality among matchings of $G$. A matching in a graph $G$ is a maximal matching if it is not properly contained in any other matching. Let $\beta_1^*(G)$ denote the minimum cardinality among maximal matchings.

Prove: For all graphs $G$, $\sigma_1(G) \leq \beta_1^*(G) \leq \beta_1(G)$. 