Combinatorics and Graph Theory Prelim, August 16, 2001

PART I. Do THREE out of the FOUR problems.

1. (a) Prove: \[ \sum_{i=0}^{k} \binom{r+i}{i} = \binom{r+k+1}{k}. \] (A combinatorial proof is preferable.)

(b) State and prove a formula (in terms of \( n \)) for the number \( D_n \) of derangements of \( n \).

2. Fifteen billiard balls can be racked into a triangular array as shown in the figure.

   (a) Assume unlimited quantities of red, white and green balls are available. Modulo the symmetry group of plane rotations of the rack, how many inequivalent color patterns of balls are possible?

   (b) Modulo the symmetry group of plane rotations of the rack, how many inequivalent color patterns of balls use 6 red, 6 white and 3 green balls?

3. Let \( r_n \) be the sequence defined by \( r_n = r_{n-1} + n \), for \( n \geq 1 \), \( r_0 = 1 \).

   (a) Find a closed form for the generating function \( g(x) = \sum_{n=0}^{\infty} r_n x^n \).

   (b) Find a closed form for the term \( r_n \) of the sequence.

4. Let \( P_n \) be the partition lattice, that is, the partitions of \( \{1, 2, \ldots, n\} \) ordered by reverse refinement.

   (a) State and prove a recursion for the Stirling numbers of the second kind, \( S(n, k) \), which count partitions of \( \{1, 2, \ldots, n\} \) into \( k \) blocks.

   (b) Describe the meet and join operations of the lattice \( P_n \).
PART II. Do THREE of the following FOUR problems. Assume all graphs are simple (no loops or multiple edges).

1. (a) Prove: Every nontrivial connected graph contains at least two vertices that are not cut-vertices.
   (b) State a useful theorem about trees that follows from (a).

2. (a) Prove: If $G$ is a graph with $n$ vertices and $\delta(G) \geq n - 2$, then $\kappa(G) = \delta(G)$. ($\kappa$ is connectivity, $\delta$ is minimum degree)
   (b) Show the statement of (a) is best possible by giving a construction, for each $n \geq 4$, of a graph with $n$ vertices, $\delta = n - 3$, and $\kappa < n - 3$.

3. Complete the statement of the theorem and prove it:
   Let $G$ be a bipartite graph with partite sets $V_1$ and $V_2$. The set $V_1$ can be matched to a subset of $V_2$ (i.e., $G$ has a matching that covers $V_1$) if and only if . . . .

4. A graph $G$ is subeulerian if it is possible to add edges to $G$ to obtain an eulerian graph (i.e., $G$ is a spanning subgraph of an eulerian graph).
   (a) Prove: If $G$ is a graph with $n$ vertices, $n$ is odd, and $n \geq 3$, then $G$ is subeulerian.
   (b) Prove: If $G$ is a complete bipartite graph with partite sets $V_1$ and $V_2$, and $|V_1|$ and $|V_2|$ are both odd, then $G$ is not subeulerian.