Lower Bounds for the Power Domination Problem

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Electric Power Network

- Thermal Power Station: 275,000V~500,000V
- Nuclear Power Station: 275,000V~500,000V
- Extra-high Voltage Substation: 154,000V
- Large Factory: 154,000V
- Transmission & Distribution Lines: 66,000V~154,000V
- Intermediate Substation: 22,000V
- Underground Distribution Lines: 22,000V
- Distributing Substation: 6,600V
- Small Plant: 100V/200V
- Services: 100V/200V
- Hydro Power Station: 275,000V~500,000V
- Large Buildings: 6,600V
- Transmission Lines: 154,000V
- Primary Substation: 66,000V
- Railroad Substation: 66,000V
- Distribution Lines: 6,600V
- Buildings: 6,600V
- Pole Transformer: 6,600V

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Electric power networks must be continuously monitored to prevent blackouts and power surges.
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Monitoring

- Electric power networks must be continuously monitored to prevent blackouts and power surges.
- Most broadly used method: placing *phase measurement units* (PMUs) at selected network locations.
- PMUs measure magnitude and phase angle of the electric wave at the locations where they are placed.
If PMU readings are synchronized via GPS it is possible to use Kirchhoff’s laws to calculate the electric waves at any network location.
The location where PMUs are placed is critical:

- the network must be fully monitored
- the cost of monitoring must be minimized
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- the network must be fully monitored
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**Definition (PMUs placing problem)**

Given an electric power network, find the locations where PMUs must be placed to monitor the entire network using the minimum number of PMUs possible.
Electrical Engineering to Graph Theory
Power Domination Problem

PMU Placing Problem $\rightarrow$ Power Domination Problem

*Engineering* $\quad$ *Graph Theory*
Power Domination Problem

PMU Placing Problem $\rightarrow$ Power Domination Problem

*Engineering* $\rightarrow$ *Graph Theory*

Power domination problem:

Given a graph, find a set of vertices, with minimum cardinality, that can *power dominate* the entire graph after the iterated application of certain rules.
Power Domination Problem

PMU Placing Problem $\rightarrow$ Power Domination Problem

Engineering $\rightarrow$ Graph Theory

Power domination problem:

Given a graph, find a set of vertices, with minimum cardinality, that can \textit{power dominate} the entire graph after the iterated application of certain rules.

The rules are defined so that placing a PMU at each vertex of a power dominating set suffices to monitor the power network.
Power Domination Rules

Rule 1: \textit{Domination rule}
A vertex power dominates itself and its neighbors.

Rule 2: \textit{Propagation rule}
If a power dominated vertex $v$ has exactly one non-power dominated neighbor $u$, then $v$ also power dominates $u$. 
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Throughout this talk, *power dominate* $=$ *monitor* $=$ *observe*
Power Domination Rules
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Lower Bounds for the Power Domination Problem
Power Domination Rules
In this work all graphs are finite, undirected, connected and have no loops and/or multiple edges.

**Definition**

\( G = (V, E) \) graph, \( S \subseteq V \), the family of sets \( M^i(S) \), \( i \in \mathbb{N} \), is defined recursively by:

- \( M^0(S) := S \)
- \( M^1(S) := N[S] = S \cup N(S) \)
- While exists \( v \in M^i(S) \) such that \( |N(v) \setminus M^i(S)| = 1 \),
  \( M^{i+1}(S) := M^i(S) \cup N(v) \).
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Power Domination Rules

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Lower Bounds for the Power Domination Problem
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\[ M^0(S) = S = \{v_3\}; \]
Power Domination Rules

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Power Domination Rules

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$M^2(S) = \{v_3, v_2, v_4, v_3, v_1, v_5, v_7\};$

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Lower Bounds for the Power Domination Problem
Power Domination Rules

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\[ M^3(S) = \{v_3, v_2, v_4, v_3, v_1, v_5, v_7, v_6\}; \]
Definition (Power dominating set)

$G = (V, E)$ graph, $S \subseteq V$ power dominating set iff $M^i(S) = V$ for some $i \in \mathbb{N}$.
Definition (Power dominating set)

\[ G = (V, E) \text{ graph, } S \subseteq V \text{ power dominating set iff } M^i(S) = V \text{ for some } i \in \mathbb{N}. \]

Definition (Power domination number)

\[ G = (V, E) \text{ graph, } \gamma_p(G) = \min \{|S| : S \subseteq V, \exists i \in \mathbb{N} : M^i(S) = V \}. \]
### Definition (Power dominating set)

Let $G = (V, E)$ be a graph, $S \subseteq V$ is a power dominating set if
and only if $M^i(S) = V$ for some $i \in \mathbb{N}$.

### Definition (Power domination number)

Let $G = (V, E)$ be a graph, then the power domination number of $G$, denoted by $\gamma_P(G)$, is defined as:

$$\gamma_P(G) = \min \{|S| : S \subseteq V, \exists i \in \mathbb{N} : M^i(S) = V\}.$$
This graph theory problem was introduced by Haynes et al. in 2002.
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**Definition (Domination number)**

Let $G = (V, E)$ be a graph, then the domination number $\gamma(G)$ is defined as:

$$\gamma(G) = \min\{|S| : N[S] = V\} = \min\{|S| : M^1(S) = V\}.$$
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**Definition (Domination number)**

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\[ \gamma(G) = \min\{|S| : N(S) = V\} = \min\{|S| : M^1(S) = V\}. \]

**Theorem (Haynes et al. 2002)**

*For every graph \( G \), \( 1 \leq \gamma_P(G) \leq \gamma(G) \).*
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**Definition (Domination number)**

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**Theorem (Haynes et al. 2002)**

*For every graph* \( G \), \( 1 \leq \gamma_P(G) \leq \gamma(G) \).

Haynes et al. also characterized the extremal graphs \( G \) such that \( \gamma_P(G) = 1 \) and those graphs \( G \) such that \( \gamma_P(G) = \gamma(G) \).
Theorem (Haynes et al. 2002)

*The power domination problem is NP-complete.*
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- *Simulations*
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Rectangular grids

Definition (Cartesian Product)

$G = (V_G, E_G)$ and $H = (V_H, E_H)$ graphs, the Cartesian product $G \Box H$ has $V(G \Box H) = V_G \times V_H$ and $E(G \Box H) = \{(g, h)(g', h') : g = g', hh' \in E_H \text{ or } h = h', gg' \in E_G\}$

Definition (Rectangular grid)

The rectangular $n \times m$ grid is $P_n \Box P_m$ where $P_n$ and $P_m$ are paths of order $n$ and $m$ respectively.
Rectangular grids
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We can extend the above construction for any \( n \geq m \geq 1 \) and prove:

\[
\gamma_P(P_n \square P_m) \leq \begin{cases} 
\left\lfloor \frac{m+1}{4} \right\rfloor & \text{if } m \equiv 4 \mod 8 \\
\left\lfloor \frac{m}{4} \right\rfloor & \text{otherwise}
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The following result requires more work:

**Theorem (Dorfling & Henning 2006)**

If $n \geq m \geq 1$, $\gamma_P(P_n \Box P_m) = \begin{cases} \left\lceil \frac{m+1}{4} \right\rceil & \text{if } m \equiv 4 \mod 8 \\ \left\lfloor \frac{m}{4} \right\rfloor & \text{otherwise} \end{cases}$
Cylinders
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Lower Bounds for the Power Domination Problem
We can extend the above construction for any $n \geq 1$ and $m \geq 3$ and prove:

$$\gamma_P(P_n \square C_m) \leq \min \left\{ \lceil \frac{m + 1}{4} \rceil, \left\lfloor \frac{n + 1}{2} \right\rfloor \right\}$$
We can extend the above construction for any $n \geq 1$ and $m \geq 3$ and prove:

$$\gamma_P(P_n \boxtimes C_m) \leq \min \left\{ \left\lceil \frac{m + 1}{4} \right\rceil, \left\lceil \frac{n + 1}{2} \right\rceil \right\}$$

The following result requires more work:

**Theorem (Barrera & DF 2011)**

*If $n \geq 1$ and $m \geq 3$, $\gamma_P(P_n \boxtimes C_m) = \min \left\{ \left\lceil \frac{m+1}{4} \right\rceil, \left\lceil \frac{n+1}{2} \right\rceil \right\}$*
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We can extend the above construction for any $m \geq n \geq 3$ and prove:

$$\gamma_P(C_n \square C_m) \leq \begin{cases} \left\lceil \frac{n+1}{2} \right\rceil & \text{if } n \equiv 2 \mod 4 \\ \left\lceil \frac{n}{2} \right\rceil & \text{otherwise} \end{cases}$$
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Consider a black/white coloring of the vertices of a graph $G$. 
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Apply the following color changing rule, until its application does not change the color of any vertex:
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*If a white vertex is the only white neighbor of a black vertex, then change its color to black.*
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If at the end of the process all vertices are black, the initial set of black vertices is called a zero-forcing set.
Consider a black/white coloring of the vertices of a graph $G$.

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The zero-forcing number of $G$, denoted as $Z(G)$, is the minimum cardinality of a zero-forcing set in $G$. 

Zero Forcing & Power Domination

Rules for power domination:

- *Domination rule*: A vertex power dominates itself and its neighbors.
- *Propagation rule*: If a power dominated vertex $v$ has exactly one non-power dominated neighbor $u$, then $v$ also power dominates $u$.

Rule for zero forcing:

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- **Domination rule**: A vertex power dominates itself and its neighbors.
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Rule for zero forcing:

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**Power domination = domination + zero forcing**
Observation:

If $S$ is a power dominating set in a graph $G$ then $N[S]$ is a zero forcing set in $G$.
If $G$ has maximum degree $\Delta$, $|N[S]| \leq |S|(\Delta + 1)$ and if $\gamma_P(G) = |S|:

$$Z(G) \leq \gamma_P(G)(\Delta + 1)$$

so

$$\left\lceil \frac{Z(G)}{\Delta + 1} \right\rceil \leq \gamma_P(G)$$
Theorem (Benson et al. 2015)

Let $G$ be a graph of maximum degree $\Delta$. Then,

$$\left\lceil \frac{Z(G)}{\Delta} \right\rceil \leq \gamma_P(G).$$
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**Sketch of the proof:** $\bigcup_{s \in S} N[s] - v_s$ where $v_s$ is an arbitrary vertex in $N(s)$, is also a zero-forcing set.
If \( \{s, t\} \) is a power dominating set, \( N[s] \cup N[t] \) is a zero forcing set but \((N[s] - \{v_s\}) \cup (N[t] - \{v_t\})\) is not.
However, \((N[s] \cup N[t]) - \{v_s\}\) is still a zero forcing set.

\[
|\left(N[s] \cup N[t]\right) - \{v_s\}| = |N[s]| + |N[t] - \{v_t\}| - 1 \leq (\Delta + 1) + \Delta - 1
\]

and we still have a zero forcing set of cardinality at most \(2\Delta\).
Theorem (Benson et al. 2015)

Let $G$ be a graph of maximum degree $\Delta$. Then, \[ \left\lceil \frac{Z(G)}{\Delta} \right\rceil \leq \gamma_P(G). \]
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If $A$ is the adjacency matrix of $G$ and $M = M(A)$ is its nullity,
\[
M \leq Z(G) \quad \text{(Barioli et al 2008)}
\]
Theorem (Benson et al. 2015)

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If $A$ is the adjacency matrix of $G$ and $M = M(A)$ is its nullity,
\[ M \leq Z(G) \]  
(Barioli et al 2008)

Corollary (Benson et al. 2015)

Let $G$ be a graph of maximum degree $\Delta$ and adjacency matrix $A$ with nullity $M = M(A)$. Then, \[ \left\lceil \frac{M}{\Delta} \right\rceil \leq \gamma_P(G). \]
Let $M_{n \times m}$ be the $n \times m$ rectangular grid, $1 \leq m \leq n$, 

\[
\frac{Z(M_{n \times m})}{4} \leq \gamma_P(M_{n \times m})
\]
Let $M_{n \times m}$ be the $n \times m$ rectangular grid, $1 \leq m \leq n$,

- $\frac{Z(M_{n \times m})}{4} \leq \gamma_P(M_{n \times m})$
- $Z(M_{n \times m}) = \min\{n, m\}$
Rectangular Grids

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- $\frac{Z(M_{n \times m})}{4} \leq \gamma_P(M_{n \times m})$
- $Z(M_{n \times m}) = \min\{n, m\}$
- $\left\lceil \frac{m}{4} \right\rceil \leq \gamma_P(M_{n \times m})$
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A simple construction shows $\gamma_P(M_{n \times m}) \leq \left\lceil \frac{m+1}{4} \right\rceil$ so,
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$$\gamma_P(M_{n \times m}) = \left\{ \begin{array}{ll} \lceil \frac{m+1}{4} \rceil & \text{if } m \equiv 4 \mod 8 \\ \lceil \frac{m}{4} \rceil & \text{otherwise} \end{array} \right.$$
Let $C_{n,m}$ be the cylinder obtained as $P_n \square C_m$, $m \geq 3$, $n \geq 1$,

- $Z(C_{n,m}) = \min\{m, 2n\}$
Let \( C_{n,m} \) be the cylinder obtained as \( P_n \Box C_m, \ m \geq 3, \ n \geq 1, \)

- \( Z(C_{n,m}) = \min\{m, 2n\} \)
- \( \Delta(C_{n,m}) = 4 \)
Let $C_{n,m}$ be the cylinder obtained as $P_n \Box C_m$, $m \geq 3$, $n \geq 1$,

- $Z(C_{n,m}) = \min \{m, 2n\}$
- $\Delta(C_{n,m}) = 4$
- $\gamma_P(C_{n,m}) \geq \min \{\lceil \frac{m}{4}\rceil, \lceil \frac{n}{2}\rceil\}$
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The construction previously seen yields

$$\gamma_P(C_{n,m}) \leq \min\left\{\left\lceil \frac{m + 1}{4} \right\rceil, \left\lceil \frac{n}{2} \right\rceil\right\}$$
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Theorem (Benson et al. 2015)

\[ \gamma_P(C_{n,m}) = \begin{cases} 
\left\lceil \frac{n^2}{2} \right\rceil & \text{if } 2n \leq m \\
\left\lceil \frac{m}{4} \right\rceil & \text{if } 2n > m \text{ and } m \not\equiv 0 \mod 4 \\
\left\lceil \frac{m + 1}{4} \right\rceil & \text{otherwise}
\end{cases} \]
\[ \min \left\{ \left\lceil \frac{m}{4} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor \right\} \leq \gamma_P(C_{n,m}) \leq \min \left\{ \left\lceil \frac{m+1}{4} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor \right\} \]

**Theorem (Benson et al. 2015)**

\[ \gamma_P(C_{n,m}) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor & \text{if } 2n \leq m \\ \left\lceil \frac{m}{4} \right\rceil & \text{if } 2n > m\text{ and } m \not\equiv 0 \mod 4 \\ \text{or } \left\lceil \frac{m+1}{4} \right\rceil & \text{otherwise} \end{cases} \]
Cylinders

\[ \min \left\{ \left\lceil \frac{m}{4} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor \right\} \leq \gamma_P(C_{n,m}) \leq \min \left\{ \left\lceil \frac{m+1}{4} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor \right\} \]

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\left\lceil \frac{m+1}{4} \right\rceil \text{ or } \left\lceil \frac{m}{4} \right\rceil & \text{otherwise}
\end{cases} \]
Let $T_{n,m}$ be the torus obtained as $C_n \Box C_m$, $m \geq n \geq 3$.

**Theorem (Benson et al. 2015)**

$$Z(T_{n,m}) = \begin{cases} 
2n - 1 & \text{if } m = n \text{ and } n \text{ is odd} \\
2n & \text{otherwise}
\end{cases}$$
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**Theorem (Benson et al. 2015)**

$$Z(T_{n,m}) = \begin{cases} 2n - 1 & \text{if } m = n \text{ and } n \text{ is odd} \\ 2n & \text{otherwise} \end{cases}$$

Obviously, $\Delta(T_{n,m}) = 4$ and $\left\lfloor \frac{2n}{4} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor$.

In addition, if $n$ is odd $\left\lfloor \frac{2n-1}{4} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor$ so we conclude

$$\gamma_P(T_{n,m}) \geq \left\lfloor \frac{n}{2} \right\rfloor.$$
From the power dominating sets for tori constructed as in the previous example,

\[ \gamma_P(T_{n,m}) \leq \begin{cases} \left\lceil \frac{n+1}{2} \right\rceil & \text{if } n \equiv 2 \mod 4 \\ \left\lceil \frac{n}{2} \right\rceil & \text{otherwise} \end{cases} \]

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**Theorem (Benson et al. 2015)**

\[ \gamma_P(T_{n,m}) \leq \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor & \text{or } \left\lceil \frac{n+1}{2} \right\rceil \text{ if } n \equiv 2 \mod 4 \\ \text{otherwise} \end{cases} \]
Lexicographic product

**Definition (Lexicographic product)**

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs, the direct product $G \ast H$ has

- $V(G \square H) = V_G \times V_H$ and
- $E(G \ast H) = \{(g, h)(g', h') : (g = g' \text{ and } hh' \in E_H) \text{ or } gg' \in E_G\}$

**Example:**

\[ A \ast a = A \ast a \]
Lexicographic product

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Example:
\[
\frac{Z(G)}{\Delta(G)} \leq \gamma_P(G) \iff Z(G) \leq \Delta(G)\gamma_P(G)
\]

• \(\gamma(G)\) is the minimum cardinality \(S \subseteq V\) s.t. \(N[S] = V\)

• \(\gamma_t(G)\) is the minimum cardinality of \(S \subseteq V\) s.t. \(N(S) = V\)

(Chang et al. 2012)
Lexicographic product

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\gamma_P(G \ast H) = \begin{cases} 
\gamma(G) & \text{if } \gamma_P(H) = 1 \\
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Lexicographic product

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- \( \deg_{G \ast H}(g, h) = (\deg_G g)|V(H)| + \deg_H h \) for any vertex \((g, h) \in V(G \ast H)\)

- \( \Delta(G \ast H) = \Delta(G)|V(H)| + \Delta(H). \)
\begin{itemize}
  \item \( \gamma_P(G \ast H) = \begin{cases} 
    \gamma(G) & \text{if } \gamma_P(H) = 1 \\
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  \item \( \Delta(G \ast H) = \Delta(G)|V(H)| + \Delta(H). \)
\end{itemize}

Thus,
\[
Z(G \ast H) \leq \begin{cases} 
  \gamma(G) \left( \Delta(G)|V(H)| + \Delta(H) \right) & \text{if } \gamma_P(H) = 1 \\
  \gamma_t(G) \left( \Delta(G)|V(H)| + \Delta(H) \right) & \text{otherwise.}
\end{cases}
\]
Assume $G$ is $d_G$-regular and $H$ is $d_H$-regular. If $\gamma_P(H) = 1$ and $\gamma(G) = 1$, then

$$Z(G \ast H) = d_G|V(H)| + d_H.$$
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If $\gamma_P(H) = 1$ and $\gamma(G) = 1$, then

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Example: If $n \geq 2$ and $m \geq 3$, $Z(K_n \ast C_m) = (n - 1)m + 2$.

(Chang et al. 2012)
Theorem (Liao 2014)

In a graph $G$ of order $n$, diameter $D$ and maximum degree $\Delta$, 
$\gamma_P(G) \geq \frac{n}{D\Delta+1}$ and this bound is best possible.

The lower bound is attained by the $16 \times m$ rectangular grid, for any integer $m \geq 2$. 

Other lower bound
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Beautiful expression for a lower bound...but it is wrong!
Theorem (Liao 2014)

In a graph $G$ of order $n$, diameter $D$ and maximum degree $\Delta$, 
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In the example below, the bound is $\frac{82}{37}$ while $\gamma_P = 2$. 

Counterexample
Given $\Delta \geq 3$, construct $H_\Delta$ with three levels of vertices as follows:

- **Level 1**: One single vertex.
- **Level 2**: $\Delta$ vertices adjacent with the vertex in level 1.
- **Level 3**: For each level 2 vertex, add $\Delta - 1$ vertices adjacent to it.
- **Add edges to make a path along level 3.**

$\gamma_P(H_\Delta) = 2$ while the bound is asymptotically $\frac{\Delta}{4}$. 
Let us look at the propagation process for a graph $G$ and a given power dominating set $S$. 

Let $M^i(S)$ denote the state of the propagation process after $i$ steps, starting with the initial state $S$. The propagation process progresses by adding vertices to the dominating set at each step, where the number of vertices added is limited by the maximum degree of the graph, $\Delta$. The propagation process is defined as follows:

- $M^0(S) = S$
- $M^i(S) \rightarrow M^{i+1}(S) \leq \Delta |S|$

The propagation process is complete when no more vertices can be added to the dominating set, which occurs when $\sum_{v \in V} \beta(v) = |V|$, where $\beta(v)$ is the propagation status of vertex $v$.

The propagation process is described by the following equations:

- $M^*(S) = M^{|V|}(S)$
- $M^+(S) = V(G)$

The number of vertices in the graph $|V|$ is bounded by the size of the initial dominating set $|S|$ plus the maximum degree $\Delta$ times the size of the initial dominating set $|S|$. This is expressed as:

$|V| \leq |S| + \Delta |S| l$
Let us look at the propagation process for a graph $G$ and a given power dominating set $S$.

In each transition $M^i(S) \rightarrow M^{i+1}(S)$ at most $\Delta |S|$ can be added.

$$\begin{align*}
M^0(S) &= S \\
M^1(S) &= N[S] \\
M^2(S) \\
M^i(S) &= V(G)
\end{align*}$$

$$|V| \leq |S| + \Delta |S|$$
Let us look at the propagation process for a graph $G$ and a given power dominating set $S$.

Let $\ell$ be the minimum positive integer $i$ such that $M^i(S) = V(G)$. Then,

$$|V| \leq |S| + \ell \Delta |S| = |S|(1 + \ell \Delta)$$

If $S$ is a *minimal* power dominating set $\gamma_P(G) = |S|$ then

$$|V| \leq \gamma_P(G)(1 + \ell \Delta)$$

so

$$\frac{|V|}{1 + \ell \Delta} \leq \gamma_P(G)$$
Let us look at the propagation process for a graph $G$ and a given power dominating set $S$. Let $\ell$ be the minimum positive integer $i$ such that $M^i(S) = V(G)$.

We know:

[For every vertex $v$ in $V$ there exists a trail joining a vertex in $S$ with $v$. The length of a trail is not upper bounded by the diameter. Except when the graph is a tree....]
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**Theorem (DF et al. 2015)**

Let $T$ be a tree with order $n$, maximum degree $\Delta$ and diameter $D$. Then,

$$\frac{n}{1 + (D - 1)\Delta} \leq \gamma_P(T)$$
Theorem (DF et al. 2015)

Let $T$ be a tree with order $n$, maximum degree $\Delta$ and diameter $D$. Then,

$$\frac{n}{1 + (D - 1)\Delta} \leq \gamma_P(T)$$

Observe that Liao’s bound for trees was

$$\frac{n}{1 + D\Delta} \leq \gamma_P(T)$$

while we obtained

$$\frac{n}{1 + (D - 1)\Delta} \leq \gamma_P(T)$$
Related problems

- Forbidden zone
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- Forbidden zone
- Fault-tolerance
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- Fault-tolerance
- Parametrized complexity
Related problems

- Forbidden zone
- Fault-tolerance
- Parametrized complexity
- Propagation number
Thank you!
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